Conformal Clifford Algebras and Image Viewpoints Orbit

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Some viewpoints of a planar object.
• Construct powerful modelings of image viewpoints and viewpoint changes

• Use these modelings to linearize the viewpoint changes by encoding them through a group of linear transformations
• **Projective geometry**: not linear $\implies$ Complication of the calculations and algorithms (see [1]).

• Consider particular viewpoint changes: dilations for instance (see [2]).

• Use a more powerful framework for modelings: **Conformal Clifford algebras**.

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1. Projective modelings: preliminary calculations

2. Clifford algebras and conformal models of $\mathbb{R}^2$ in the Minkowski space:
   - Definitions and examples
   - Horospheres

3. Conformal modelings: two approaches
   - A conformal image: mapping defined on a horosphere
   - A conformal viewpoint change: horospheres change
   - The set of all the viewpoint changes:
     1. approach 1 $\Rightarrow$ group
     2. approach 2 $\Rightarrow$ groupoid
   - Approach 1 versus approach 2
**Figure:** The extrinsic parameters \((\theta, \phi, \psi, \lambda, t_1, t_2)\) of the camera (where \(t_1 = t_2 = 0\))

Frontal viewpoint \(\iff \theta = 0\)

A projective image is a mapping:

\[ l_\theta : h_\theta (\mathbb{R}^2) \subset \mathbb{P}^2 \mathbb{R} \rightarrow \mathbb{R}. \]  \hfill (1)

where \( h_\theta \) is a homography of the real projective plane \( \mathbb{P}^2 \mathbb{R} \) called perspective distortion that satisfies:

\[ h_\theta [\pi(x_1, x_2, 1)] = \pi[M_\theta.(x_1, x_2, 1)], \]  \hfill (2)

and

\[ M_\theta = \begin{pmatrix} \cos \theta & 0 & 0 \\ 0 & 1 & 0 \\ -\sin \theta & 0 & 1 \end{pmatrix}. \]  \hfill (3)

\( \theta \in [0, \pi/2[ \) is the latitude angle of the camera and \( h_\theta (\mathbb{R}^2) \) is a perspective plane and \( \pi : \mathbb{R}^3 - \{0\} \rightarrow \mathbb{P}^2 \mathbb{R} \) is the canonical projection.
**Figure:** A frontal image of a planar object (left), a slanted view of the object for \( \theta = \pi/6 \) (middle) and a representation of the spacial domain of the distorted image (right).
We denote by $\mathbb{R}^{p,q}$ the space $\mathbb{R}^n$ equipped with the quadratic form $Q$ of signature $(p, q)$ with $p + q = n$.

The Clifford algebra (or geometric algebra) $\mathbb{R}_{p,q}$ of $\mathbb{R}^{p,q}$ is an associative algebra that contains the vectors of $\mathbb{R}^{p,q}$ and the scalars of $\mathbb{R}$ (see [1]).

The Clifford multiplication that defines $\mathbb{R}_{p,q}$ as a unitary ring is called geometric product.

If $\{e_1, e_2, ..., e_n\}$ is a basis of $\mathbb{R}^{p,q}$, then $\mathbb{R}_{p,q}$ is the algebra generated by the vectors $e_i$. It is of dimension $2^n$ and admits the set

$$\{1, e_{i_1} ... e_{i_k}, \ i_1 < ... < i_k, \ k \in \{1, ..., n\}\}$$

as basis.

Clifford algebras

Spin group

Examples:
- \( \mathbb{R}_{0,1} \) is isomorphic to the commutative field of complex numbers
- \( \mathbb{R}_{0,2} \) is isomorphic to the non-commutative field of quaternions

The Spin group

It is the set of the following elements of \( \mathbb{R}_{p,q} \)

\[
\text{Spin}(p, q) := \{ \sigma = \prod_{i=1}^{2k} u_i, \ |Q(u_i)| = 1 \} \quad (5)
\]
Consider the isotropic basis \( \{e_\infty, \alpha, e_0, \alpha\} \) of \( \mathbb{R}^{1,1} \) (for \( \alpha > 0 \)):

\[
e_\infty, \alpha = (\alpha, \alpha) \quad \text{and} \quad e_0, \alpha = \frac{1}{2} \left( -\frac{1}{\alpha}, \frac{1}{\alpha} \right)
\]  

(6)

satisfying \( e_\infty, \alpha \cdot e_0, \alpha = -1 \).

The space \( \mathbb{R}^{3,1} \) is decomposed into a conformal split

\[
\mathbb{R}^{3,1} = \mathbb{R}^2 \oplus \mathbb{R}^{1,1}
\]  

(7)

where \( \{e_1, e_2\} \) is an orthonormal basis of \( \mathbb{R}^2 \). Thus, \( \{e_1, e_2, e_\infty, \alpha, e_0, \alpha\} \) is a basis of \( \mathbb{R}^{3,1} \).

The horosphere $H_\alpha$ associated with the basis $\{e_\infty, \alpha, e_0, \alpha\}$ is the set of normalized isotropic vectors of $\mathbb{R}^{3,1}$:

$$H_\alpha = \{X \in \mathbb{R}^{3,1}; \ X^2 = 0 \text{ and } X \cdot e_\infty, \alpha = -1\}. \quad (8)$$

More precisely, $H_\alpha = \varphi_\alpha(\mathbb{R}^2)$ where $\varphi_\alpha$ is a global parametrization

$$\varphi_\alpha : \mathbb{R}^2 \longrightarrow H_\alpha \quad (9)$$

that sends $x \in \mathbb{R}^2$ to

$$X_\alpha = \varphi_\alpha(x) = x + \frac{1}{2}x^2 e_\infty, \alpha + e_0, \alpha. \quad (10)$$
Conformal modelings : approach 1

Conformal image

Proposition

An image is the data of a one parameter horosphere $H_{\alpha\theta}$ encoding the latitude angle $\theta$ and a mapping $\overline{I}_{\alpha\theta}$ defined on $H_{\alpha\theta}$ by

$$\overline{I}_{\alpha\theta} : \{ H_{\alpha\theta} : \varphi_{\alpha\theta} \} \rightarrow \mathbb{R}$$

$$X_{\alpha\theta} = \varphi_{\alpha\theta}(x) \quad \mapsto \quad I_\theta \circ h_\theta \circ \varphi_{\alpha\theta}^{-1}(X_{\alpha\theta})$$

$$= \quad I_\theta \circ h_\theta(x),$$

where $I_\theta$ is the corresponding projective image.
Conformal modelings : approach 1

Conformal viewpoint change

**Figure**: First line: an image \( l_{\theta_1} \) (right) and its associated frontal image \( l^1_0 \) (left). Second line: an image \( l_{\theta_2} \) (right) and its associated frontal image \( l^2_0 \) (left).
The group that models the viewpoint changes is the sub-group $C_0(3, 1)$ of the group $C(3, 1)$ of the linear conformal transformations of $\mathbb{R}^{3,1}$ satisfying:

$$\{ F|\{H_{\alpha \theta}, \varphi_{\alpha \theta}\} : F \in C_0(3, 1) \text{ et } \alpha \theta > 0 \}$$

$$= \{ \tilde{F} = \varphi_{\alpha \theta_2} \circ f \circ \varphi_{\alpha \theta_1}^{-1} , f \text{ similitude, } \alpha_{\theta_i} > 0, \}$$

Technical calculations gives:

$$F \in C_0(3, 1) \iff F = F_{\alpha} \circ F_{\sigma}$$

(13)

where $\bar{\alpha} > 0$ and $\sigma$ is a spinor encoding the similarities and

$$F_{\bar{\alpha}} \circ \varphi_{\alpha \theta} = \varphi_{\bar{\alpha} \alpha \theta}$$

(14)

$$F_{\sigma} \circ \varphi_{\alpha \theta} = \varphi_{\alpha \theta} \circ f_{\alpha \theta}$$

(15)
Aim: Introduce more natural modeling of the latitude and the zoom of the camera.

Let $e_\infty = (1, 1)$ and $e_0 = 1/2(-1, 1)$ be the two isotropic vectors of the standard conformal model of $\mathbb{R}^2$ ($\alpha = 1$).

Main idea

The effect of the latitude $\theta \iff$ applying on $e_\infty$ a rotation $\rho_\theta$ of angle $\theta$.

The effect of the zoom $\lambda > 0 \iff$ applying on $e_\infty$ a dilatation of ratio $\lambda$. 
We propose then to introduce the vectors $\lambda e_{\infty, \theta}$ and $\lambda^{-1} e_{0, \theta}$ where:

$$
e_{\infty, \theta} = \rho_\theta(e_{\infty}) = \begin{pmatrix} \cos \theta - \sin \theta \\ \sin \theta + \cos \theta \end{pmatrix} \quad e_{0, \theta} = \rho_\theta(e_0) = -\frac{1}{2} \begin{pmatrix} \cos \theta + \sin \theta \\ \sin \theta - \cos \theta \end{pmatrix}
$$

(16)

isotropic for the new metric

$$G_\theta = \rho_\theta \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \rho_\theta^{-1} = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}
$$

(17)

and satisfying $e_{\infty, \theta} \cdot e_{0, \theta} = -1$. 
For every $\theta$, the space is then $\mathbb{R}^4_\theta = (\mathbb{R}^4, Q_\theta)$ where

$$Q_\theta = id \oplus G_\theta$$

(18)

**Definition**

A generalized conformal representation of the Euclidean plane is the data of a two-parameter horosphere:

$$H(\lambda, \theta) = \{X \in \mathbb{R}^4_\theta, \quad X^2 = 0, \quad X \cdot \lambda e_{\infty, \theta} = -1\}$$

(19)

associated with the embedding $\varphi_{\lambda, \theta}$ of $\mathbb{R}^2$ into $\mathbb{R}^4_\theta$:

$$\varphi_{\lambda, \theta}(x) = x + \frac{1}{2}x^2\lambda e_{\infty, \theta} + \lambda^{-1}e_{0, \theta}$$

(20)
An image is the data of a generalized conformal model \((H(\lambda, \theta), \varphi_{\lambda, \theta})\) and a mapping:

\[
\overline{I}_{\lambda, \theta} : (H(\lambda, \theta), \varphi_{\lambda, \theta}) \rightarrow \mathbb{R} \tag{21}
\]

\[
X_{\lambda, \theta} = \varphi_{\lambda, \theta}(x) \quad \mapsto \quad I_{\lambda, \theta} \circ h_\theta \circ D_{\lambda^{-1}} \circ \varphi_{\lambda, \theta}^{-1}(X_{\lambda, \theta})
\]

\[
= I_{\lambda, \theta} \circ h_\theta \circ D_{\lambda^{-1}}(x),
\]

where \(I_{\lambda, \theta}\) is the corresponding projective image.
**Groupoïd:**

- It is a category whose morphisms are isomorphisms
  \[\Rightarrow\] the set of morphisms of a groupoïd generalize the notion of group.

**Generalization of the notion of group action:**

Consider a group $G$ acting on a set $X$ by

$$G \times X \rightarrow X \quad (22)$$

To this action corresponds a groupoïd:

- the objects are the elements of $X$
- the morphisms from $x$ to $y$ are the elements of $G$ that send $x$ to $y$
The groupoïd \( \mathcal{PV} \) of the viewpoints and viewpoint changes:

- The objects are \((H(\lambda, \theta), \varphi_{\lambda, \theta}) \iff \text{representing the viewpoints}\)
- The morphisms encode \text{the viewpoint changes} and are the compositions of the basic diagrams:
  1. \((\sigma_t, T_t)\) where \(\sigma_t = 1 - \frac{1}{2} t \lambda e_{\infty, \theta}\)
  2. \((\sigma_\gamma, R_\gamma)\) where \(\sigma_\gamma = \exp[-\frac{\gamma}{2} e_1 \land e_2]\)
  3. \((\sigma_\delta, id)\) where \(\sigma_\delta = \exp[-\frac{1}{2} E_\theta \ln \delta]\) and \(E_\theta = e_{\infty, \theta} \land e_{0, \theta}\)
  4. \((\rho_\varphi, id)\)

\[
\rho_\varphi = id \oplus \begin{pmatrix}
\cos \varphi & -\sin \varphi \\
\sin \varphi & \cos \varphi
\end{pmatrix}
\]

(23)

in the canonical basis \(\{e_1, e_2, e_+, e_-\}\) of \(\mathbb{R}^4\).
Conformal modelings : approach 2

Groupoïd of the conformal viewpoint changes

The basic morphisms:

\[
\begin{align*}
H(\lambda, \theta) & \xrightarrow{\sigma_t} H(\lambda, \theta) \\
\varphi_{\lambda, \theta} & \xleftarrow{R^2} \xrightarrow{T_t} \varphi_{\lambda, \theta} \\
H(\lambda, \theta) & \xrightarrow{\sigma_\gamma} H(\lambda, \theta) \\
\varphi_{\lambda, \theta} & \xleftarrow{R^2} \xrightarrow{R_\gamma} \varphi_{\lambda, \theta} \\
H(\lambda, \theta) & \xrightarrow{\sigma_\delta} H(\delta \lambda, \theta) \\
\varphi_{\lambda, \theta} & \xleftarrow{R^2} \xrightarrow{id} \varphi_{\delta \lambda, \theta} \\
H(\lambda, \theta) & \xrightarrow{\rho_\varphi} H(\lambda, \theta + \varphi) \\
\varphi_{\lambda, \theta} & \xleftarrow{R^2} \xrightarrow{id} \varphi_{\lambda, \theta + \varphi}
\end{align*}
\]
### Approach 1 versus approach 2

<table>
<thead>
<tr>
<th></th>
<th>THE GROUP $C_0(3, 1)$</th>
<th>THE GROUPOÏD $\mathcal{PV}$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Domain of $\theta$</strong></td>
<td>$]0, \frac{\pi}{2}[$</td>
<td>$S^1$</td>
</tr>
<tr>
<td></td>
<td>$\theta \neq 0$</td>
<td>all values</td>
</tr>
<tr>
<td><strong>Modeling of $\theta$</strong></td>
<td>bijection $\theta \mapsto \alpha_\theta$</td>
<td>more natural action of $\theta = \text{rotation of } e_\infty$</td>
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<tr>
<td></td>
<td>$\alpha_\theta = \cot \theta$</td>
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<tr>
<td><strong>Modeling of the dilations of ratio $\delta$</strong></td>
<td>transformations of $\mathbb{R}^2$</td>
<td>zoom change</td>
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<tr>
<td></td>
<td>$x \mapsto \delta x$</td>
<td>$\lambda e_\infty \mapsto \delta \lambda e_\infty$</td>
</tr>
<tr>
<td><strong>Modeling of the viewpoint changes</strong></td>
<td>sub-group of $C(3, 1)$</td>
<td>morphisms of groupoïd $\mathcal{PV}$</td>
</tr>
</tbody>
</table>
Modelings in the projective geometry

Conformal model of $\mathbb{R}^2$ in the Minkowski space

Conformal modelings in computer vision: approach 1

1. constant metric (Minkowski)
2. embedding the domain of the projective image in $\mathbb{R}^{3,1}$
3. an image is defined on a one-parameter horosphere $H_{\alpha, \theta}$
4. construction of a group of linear conformal transformations $\mathbb{R}^{3,1}$ encoding the viewpoint changes

Conformal modelings in computer vision: approche 2

1. generalized conformal model: metric change for every $\theta$
2. an image is defined on a two-parameter horosphere $H_{\lambda, \theta}$
3. construction of a groupoid whose morphisms encode the viewpoint changes
Calculations of viewpoint invariants by the action of the group (or the groupoid') of the viewpoint changes on the set of conformal images.

Practical implementation of algorithms in computer vision for the search of viewpoint invariants.
THANK YOU FOR YOUR ATTENTION!
Covariant detector by the action of $C_0(3,1)$ on $\bar{I}$

It is a functional $\Psi : \bar{I} \times C_0(3,1) \rightarrow \mathbb{R}$ differentiable according to a parametrization of $C_0(3,1)$ and satisfying

1. **Existence of the Canonical Element**: for all $\bar{I}_{\alpha \theta} \in \bar{I}$, there exists $\bar{F}(\bar{I}_{\alpha \theta}) \in C_0(3,1)$ such that
   \[ \nabla \Psi(\bar{I}_{\alpha \theta}, \bar{F}(\bar{I}_{\alpha \theta})) = 0. \]  
   \hspace{1cm} (24)
   This element $\bar{F}(\bar{I}_{\alpha \theta})$ of the group is called canonical element of $\bar{I}_{\alpha \theta}$.

2. **Transversality Condition**: the Hessian matrix of $\Psi$ is non-degenerate i.e.
   \[ \det[H_{\Psi}(\bar{I}_{\alpha \theta}, \bar{F}(\bar{I}_{\alpha \theta}))] \neq 0, \]  
   \hspace{1cm} (25)
   for all $(\bar{I}_{\alpha \theta}, \bar{F}(\bar{I}_{\alpha \theta}))$ satisfying (24),

3. **Covariance Condition**: if $\nabla \Psi(\bar{I}_{\alpha \theta}, \bar{F}(\bar{I}_{\alpha \theta})) = 0$ then
   \[ \nabla \Psi(F \ast \bar{I}_{\alpha \theta}, F \circ \bar{F}(\bar{I}_{\alpha \theta})) = 0 \]  
   \hspace{1cm} (26)
   for all $F \in C_0(3,1)$.

\[ \bar{I} \] is the set of conformal images $\bar{I}_{\alpha \theta}$ and $\ast$ denotes the action of $C_0(3,1)$ on $\bar{I}$. 
It is a functional $\Phi$ defined on $\mathbb{I}$ by:

$$
\Phi(\bar{I}_{\alpha\theta}) = [F(\bar{I}_{\alpha\theta})]^{-1} \ast \bar{I}_{\alpha\theta}.
$$

(27)

It is a complete invariant by the action $\ast$ of $C_0(3,1)$ on $\mathbb{I}$:

$$
\Phi(\bar{I}_{\alpha\theta_1}) = \Phi(\bar{I}_{\alpha\theta_2}) \iff \exists F \in C_0(3,1) \text{ tel que } F \ast \bar{I}_{\alpha\theta_1} = \bar{I}_{\alpha\theta_2}.
$$

(28)