The Riemann Mapping Theorem For Harmonic Mappings

Abdallah Lyzzaik
Department of Mathematics
American University of Beirut

Abstract

Let Ω be a bounded simply connected domain containing a point w_0 and having a locally connected boundary, and let ω be an analytic function of the open unit disc $\mathbf{D} = \{\mathbf{z} : |\mathbf{z}| < \mathbf{1}\}$ satisfying $|\omega| < 1$. It is known that there exists a one-to-one planar harmonic map of $\mathbf{D}$ normalized by $f(0) = w_0$ and $f_z(0) > 0$ and which maps $\mathbf{D}$ into Ω such that (i) the unrestricted limit $f^*(e^{it}) = \lim_{z \to e^{it}} f(z)$ exists and belongs to $\partial\Omega$ for all but a countable subset E of the unit circle $\mathbf{T} = \partial\mathbf{D}$, (ii) f^* is a continuous function on $\mathbf{T} \setminus \mathbf{E}$ such that for every $e^{is} \in E$ the one-sided limits $\lim_{t \to s^+} f^*(e^{it})$ and $\lim_{t \to s^-} f^*(e^{it})$ exist, belong to $\partial\Omega$, and are distinct, and (iii) the cluster set of f at $e^{is} \in E$ is the straight line segment joining the one-sided limits $\lim_{t \to s^+} f^*(e^{it})$ and $\lim_{t \to s^-} f^*(e^{it})$. It has been a major question, since the thirties of the twentieth century, to establish the uniqueness of f. The purpose of this talk is to survey the main results obtained in the course of resolving this question.