NDU Math Competition for High School students

Notre Dame University, Lebanon

Department of Mathematics and Statistics

You have 17 Questions

March 12, 2018

Question 1

Let $x \in \mathbb{R}$ such that $x + \frac{1}{x} \in \mathbb{Z}$. Show that

$$x^2 + \frac{1}{x^2} \in \mathbb{Z}.$$

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You have 5 minutes

Question 2

A car travels from a to b at an average speed of 50 km/hour. At what average speed would it have to travel from b to a to average 60 km/hour for the whole trip?

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You have 5 minutes

Question 3

Let f be a real and continuous function satisfying $f(2x) = 4f(x)$ for all x. If $\int_0^1 f(x)\ dx = 1$, find $\int_1^2 f(x)dx$.

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You have 5 minutes

Question 4

Evaluate the following integral

$$I = \int_0^a \frac{f(x)dx}{f(x) + f(a-x)}$$

reset

You have 5 minutes

Question 5

How many digits are in the number 20^{15} ?

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You have 5 minutes

Question 6

Find the last 3 digits of the sum

$$1 + 11 + 111 + 1111 + \cdots + 11111 \cdots 11111,$$

where the last number has 99 ones.

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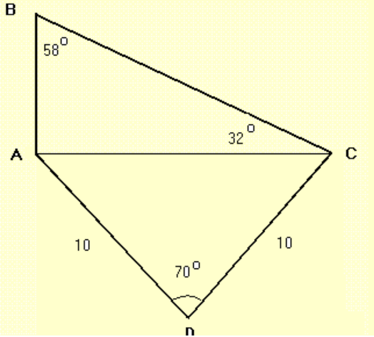
You have 5 minutes

Question 7

Find the length of side AB in the figure below.



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You have 5 minutes

Question 8

Find all values of $a \in \mathbb{R}$ for which the following system has infinitely many solutions:

$$\begin{cases} x & + & (2a+1)y & = & 0 \\ 4x & - & (a-13)y & = & 0 \end{cases}$$

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You have 5 minutes

Question 9

Let $p_a(x) = x^2 - 2x + a^2$ where $a \in \mathbb{R}$ is a real parameter. Define the set

$$S = \{x \in \mathbb{R}; p_a(x) = 0, \text{ for some } a \in \mathbb{R}\}$$

Determine the set S explicitly.

reset

You have 5 minutes

Question 10

There are 4 urns and 10 identical balls. Determine the total number of possibilities one can distribute these 10 balls among these 4 urns. So for instance,

urn 1: 4 balls

urn 2: 3 balls

urn 3: 3 balls

urn 4: 0 balls

This would correspond to one possibility. What is the total number of possibilities?

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You have 8 minutes

Question 11

Let $f(x) = 2x - e^x$. Find the maximum value of $f(x)$ on the interval $[0, 1]$. At which value(s) of x is it attained?

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You have 2 minutes

Question 12

Let $f(x) = \sqrt[3]{2x+1}$ and $g(x) = x + b$ both defined on $]-\infty, +\infty[$. If $h(x) = f \circ g(x)$ passes by the point $(1, 2)$, calculate b.

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You have 3 minutes

Question 13

For $\frac{\pi}{2} \leq x \leq \pi$, simplify the expression

$$2(1 - \sin^2 x)\frac{\sqrt{1 - \cos^2 x}}{\sin x} + \frac{\sqrt{1 - \sin^2 x}}{\cos x}$$

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You have 5 minutes

Question 14

The area of a triangle is bounded by the lines $y = 2x$, $y = 0$, and $y = -0.5x + k$ is $80\ cm^2$. Solve for $k > 0$.

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You have 7 minutes

Question 15

Let P be a polynomial such that $P(x^2+1) = -2x^4 + 5x^2 + 6$. Find $P(-x^2+3)$.

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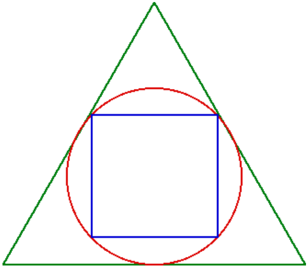
You have 5 minutes

Question 16

A square is inscribed in a circle that is inscribed in an equilateral triangle of side $a = 2$. Calculate the area of the square.



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You have 8 minutes

Question 17

If $\sin\theta = x\cos\phi$ and $\cos\theta = y\sin\phi$, what is the value of $\cos^2\phi$ in terms of x and y?

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You have 3 minutes