Rescaling and Parallel Time Integration for systems of Order
Differential Equations

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Abstract
We consider a system of time-dependent Ordinary Differential Equations (ODE’s), where one seeks
\[ Y : [0, \infty) \to \mathbb{R}^k \]
\[ \frac{dY}{dt} = F(Y), \quad 0 < t \leq T \leq \infty, \quad Y(0) = Y_0. \]  

(1)

Numerical (time) integration procedures to obtain approximate solutions to (1) are inherently sequential. All ODE’s solvers are based on algorithms that “advance” a numerical process from a time \( t_k \) to \( t_{k+1} \), where the sequence \( \{t_k\} \) is usually a “refined” uniform or adaptive subdivision of \([0,T]\). Time-Parallel methods become interesting when one seeks to reduce the excessively high computational time of (1), where the time of existence is of very large order \( T \approx \infty \). Time Parallelization is based on iterative Predictor-Corrector Schemes. Its success depends on appropriate predictions of the solution at the beginning of each “time slice” of some coarse grid \( \{[T_{n-1}, T_n]\} \), whereas \([0,T] = \bigcup_{n=1}^{N} [T_{n-1}, T_n] \). Good predictions would automatically reduce the total number of iterations \( N_{it} \), \((N_{it} \leq N)\). A key performance indicator as to the high performance of the method should verify the criterion: \( N_{it} \ll N \), i.e. the number of iterations should be much less than the number of slices.

On the basis of a multi-scaling technique that rescales both the variables \( Y \) and \( t \) on every slice, we introduce a concept of similarity which leads to a “ratio property” in the case when

\[ (F(Y))_i = \sum_j a_{ij} Y_i^{k_{i,j}} Y_j^{l_{i,j}}. \]

Such situations occur when solving diffusion reaction problems an also Lotka-Voltera predators-preys logistics models.

In this paper, these concepts are applied to a second order initial value model:

\[ y'' - |y'|^{q-1} y' + |y|^{p-1} y = 0, \quad t > 0, \quad y(0) = y_{1,0}, \quad y'(0) = y_{2,0}. \]  

(2)

This ODE describes the motion of a membrane. Specifically, when

\[ p \leq q \leq \frac{2p}{p+1} \quad \text{and} \quad p < 1, \]  

(3)

the existence of the solution is global with a non-oscillatory blow-up at \( \infty \), i.e.:

\[ \lim_{t \to \infty} |y(t)| = \lim_{t \to \infty} |y'(t)| = \infty, \]  

(4)

the roots of \( y \) and \( y' \) being both countably infinite sets.

We show through slices rescaling, that similarity can be obtained with a ratio property leading to an efficient parallel time solver for (2).