Cohomology of Algebraic Plane Curves

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    - Koszul Complex
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Notations

- $S = \mathbb{C}[x, y, z] = \oplus_{r \geq 0} S_r$, and $f \in S_N$.
- $J_f = (f_x, f_y, f_z)$ the *Jacobian ideal* of f.
- The graded Milnor algebra of f is given by:

$$M(f) = S/J_f = \oplus_{r \geq 0} M(f)_r.$$

- For any graded module $M = \oplus_{s \geq s_0} M_s$ over a $\mathbb{C}$-algebra of finite type, define the *Poincaré Series* by

$$P(M)(t) = \sum_{s \geq s_0} (\dim_{\mathbb{C}} M_s) t^s.$$

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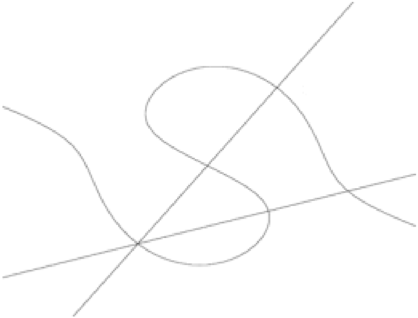
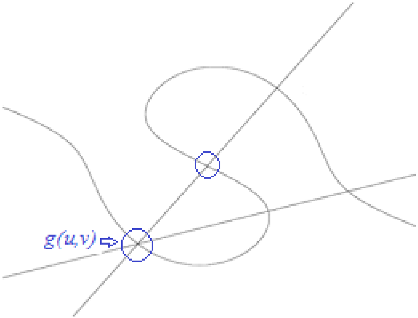
Let $C \subset \mathbb{P}^2 : f = 0$ be a curve having an isolated singularity at a point P,



Let $C \subset \mathbb{P}^2 : f = 0$ be a curve having an isolated singularity at a point P, and let $g(u, v)$ the local equation of f at P, then



Let $C \subset \mathbb{P}^2 : f = 0$ be a curve having an isolated singularity at a point P, and let $g(u, v)$ the local equation of f at P, then

- the *Milnor number* of f at P is given by

$$\mu(C, P) = \dim_{\mathbb{C}} \frac{O_P}{J_g}.$$

- the *Tjurina number* of f at P is given by

$$\tau(C, P) = \dim_{\mathbb{C}} \frac{O_P}{(g, J_g)}.$$

- The Milnor (Tjurina) number of the curve C is the sum of the Milnor (Tjurina) numbers of all the singularities of C.

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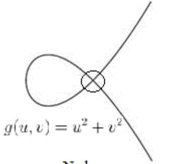
Example

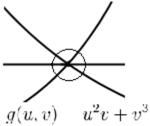
- Node or A_1 singularity
  $\mu(C, P) = \tau(C, P) = 1$.



Node

Example

- Ordinary triple point
  or D_4 singularity
  $\mu(C, P) = \tau(C, P) = 4$.



Triple Point

- Let $r(C, P)$ be the number of irreducible branches of the germ (C, P), the δ-invariant of C at the point P is defined by

$$\delta(C, P) = \frac{1}{2}(\mu(C, P) + r(C, P) - 1).$$

Example

- For a node, $r = 2$, and hence the $\delta = 1$.
- For an ordinary triple point $r = 3$, and hence the $\delta = 3$.

- The genus g of C is given by

$$g = \frac{(N-1)(N-2)}{2} - \sum_k \delta(C, P_k).$$

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- Let $AR(f) = \oplus_{r\geq 0} AR(f)_r$ be a graded S-module, where $AR(f)_r = \{(a, b, c) \in S_r^3 : af_x + bf_y + cf_z = 0\}$. $KR(f) \subset AR(f)$ the submodule of Koszul relations or trivial relations spanned by the relations of the form $(f_i)f_j + (-f_j)f_i = 0$. The quotient module $ER(f) = AR(f)/KR(f)$ is called the module of nontrivial syzygies or essential relations.

Goals

- Relation between the Milnor algebra and the singularities of the curve $C \subset \mathbb{P}^2 : f = 0$.
- Relation between the Hodge theory of the complement $U = \mathbb{P}^2 \setminus C$ and the singularities of C.

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Koszul Complex

Let $\Omega^p = \{\sum_I c_I dx_{i_1} \wedge \cdots \wedge dx_{i_p}\}$,
where $I = (i_1, \cdots, i_p)$, with $x_{i_j} \in \{x, y, z\}$, and $c_I \in \mathbb{C}[x, y, z]$.

For homogeneous polynomials f_0, f_1, f_2, the Koszul complex is given by

$$K^*(f_0, f_1, f_2) : 0 \to \Omega^0 \xrightarrow{\omega\wedge} \Omega^1 \xrightarrow{\omega\wedge} \Omega^2 \xrightarrow{\omega\wedge} \Omega^3 \to 0$$

where $\omega = f_0 dx + f_1 dy + f_2 dz$.

Example

Let $f \in S_N$, f_x, f_y, f_z the partial derivatives of f, then,

$$K^*(\mathbf{f}) = K^*(f_x, f_y, f_z) : 0 \to \Omega^0 \xrightarrow{\omega\wedge} \Omega^1 \xrightarrow{\omega\wedge} \Omega^2 \xrightarrow{\omega\wedge} \Omega^3 \to 0$$

with $\omega = df = f_x dx + f_y dy + f_z dz$, is the *Koszul Complex* of the partial derivatives of f.

Remark

$im(\Omega^2 \xrightarrow{\omega\wedge} \Omega^3) = J_f$, and therefore $H^3(K^*(\mathbf{f})) = M(f)$, and $H^2(K^*(\mathbf{f})) = ER(f)$, in particular $H^3(K^*(\mathbf{f}))_{k+3} = M(f)_k$, and $H^2(K^*(\mathbf{f}))_{k+2} = ER(f)_k$ for $k \geq 0$.

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Koszul Complex and Singularities

Proposition (Kyoji Saito, 1974)

Let $\Sigma = V(f_x, f_y, f_z) \subset \mathbb{P}^2$ *then,*

$$H^{3-k}(K^*(\boldsymbol{f})) = 0 \text{ for } k > \dim(\Sigma) + 1,$$

where $K^*(\boldsymbol{f})$ *is the Koszul complex of the partial derivatives of* f*.*

Smooth Case

$f \in S_N$, $C \subset \mathbb{P}^2 : f = 0$ a smooth curve, then $H^{3-k}(K^*(\mathbf{f})) = 0$ for all $k > 0$, and the Poincaré series is completely determined, namely

$$P(M(f))(t) = t^{-3} P(H^3(K^*(\mathbf{f})))(t) = \frac{(1 - t^{N-1})^3}{(1-t)^3}.$$

Remark

The Poincaré series depends only of the degree of f, and it is a polynomial of degree $3N - 6$ with the property $M(f)_k = M(f)_{3N-6-k}$.

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Singular Case

If $C \subset \mathbb{P}^2$ has only isolated singularities, then $H^{3-k}(K^*(\mathbf{f})) = 0$ for $k > 1$, and the nonzero cohomology groups are related as follows:

$$t^N P(H^2(K^*(\mathbf{f})))(t) = P(H^3(K^*(\mathbf{f}))(t) - t^3 \frac{(1 - t^{N-1})^3}{(1 - t)^3}.$$

Proposition (Choudary, Dimca, 1994)

The sequence $\dim M(f)_k$ *decreases for* $k \geq 2(N-2)$ *and becomes constant for* $k \geq 3N-5$*. More precisely, for* $k \geq 3N-5$, $\dim M(f)_k = \tau(C)$.

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Definition

(i) $ct(C) = max\{q : \dim M(f)_k = \dim M(f_s)_k \text{ for all } k \leq q\}$, with $f_s \in S_N$ such that C_s: $f_s = 0$ is a smooth curve in $\mathbb{P}^2$.

(ii) $st(C) = min\{q : \dim M(f)_k = \tau(C) \text{ for all } k \geq q\}$.

(iii) $mdr(C) = min\{q : ER(f)_q \neq 0\}$.

We have:

$ct(C) = mdr(C) + N - 2$,
$N - 2 \leq ct(C) \leq 3(N - 2)$,
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Nodal Curves

Proposition (Dimca, Sticlaru, 2011)

Let $C : f = 0$ be a nodal curve of degree N in $\mathbb{P}^2$. Then one has $ct(C) \geq 2N - 4$, and

$$\dim M(f)_{2N-3} = n(C) + \sum_{j=1}^{r} g_j$$

where $n(C) = \tau(C)$ is the total number of nodes of C and g_j are the genera of the irreducible components C_j of C whose number is r.

Example

Let $C : f = x(x^3 + y^3 + z^3) = 0$.
$\dim M(f)_{2N-3} = 3 + 1 = 4$, $st(C) \leq 3N - 5 = 7$ and $ct(C) \geq 2N - 4 = 4$. By Singular,

$$P(M(f))(t) = 1 + 3t + 6t^2 + 7t^3 + 6t^4 + 4t^5 + 3(t^6 + t^7 + \cdots),$$

and hence $ct(C) = 4$ and $st(C) = 6$.

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If C is a rational nodal curve, then the Poincaré series of the Milnor algebra is completely determined, and $st(C) \leq 2N - 3$ unless C is a generic line arrangement then $st(C) = 2N - 4$.

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Example

Let $C : f = xyz(x + y + z) = 0$. C has 6 nodes, $P(M(f))$ is all determined and we have $st(C) = 2N - 4 = 4$ and $ct(C) \geq 4$. Therefore

$$P(M(f))(t) = 1 + 3t + 6t^2 + 7t^3 + 6(t^4 + t^5 + \cdots),$$

which implies that $ct(C) = 4$.

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Let $C : f = x^{N-1}y + z^N = 0$, then $xf_x - (N - 1)yf_y = 0$. Therefore $mdr(C) = 1$, and $ct(C) = N - 1 < 2N - 4$.

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Question: What happens in the general case?

Generalization of these results to curves with ordinary double and triple points

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Curves with Ordinary Double and Triple Points

Theorem

Let C be a plane curve in $\mathbb{P}^2$ given by $f = 0$, $f \in S_N$ with n nodes (A_1) and t triple points (D_4), then $\tau = n + 4t$. Let $C = \bigcup_{j=1,r} C_j$, $U = \mathbb{P}^2 \setminus C$, and $g_j = g(C_j)$.

(A) $0 \leq \dim M(f)_{2N-3} - \tau \leq \sum_{j=1}^{r} g_j$. *In particular,*

(i) *If all $g_i = 0$, one has $\dim M(f)_{2N-3} = \tau$, i.e. $st(C) \leq 2N - 3$.*

(ii) $\dim M(f)_{2N-3} - \tau = \sum_{j=1}^{r} g_j$ *if and only if $H^2(U)$ satisfies $F^2H^2(U) = P^2H^2(U)$.*

(B) *$max(r - 1 + t - \sum_{j=1}^{r} g_j, r - 1) \leq \dim ER(f)_{N-2} \leq r - 1 + t$. In particular, $\dim ER(f)_{N-2} = r - 1 + t$ if $g_j = 0$ for all j.*

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Curves with Ordinary Double and Triple Points

Theorem

Let C be a plane curve in $\mathbb{P}^2$ given by $f = 0$, $f \in S_N$ with n nodes (A_1) and t triple points (D_4), then $\tau = n + 4t$. Let $C = \bigcup_{j=1,r} C_j$, $U = \mathbb{P}^2 \setminus C$, and $g_j = g(C_j)$.

(A) $0 \leq \dim M(f)_{2N-3} - \tau \leq \sum_{j=1}^{r} g_j$. *In particular,*

- (i) *If all $g_i = 0$, one has $\dim M(f)_{2N-3} = \tau$, i.e. $st(C) \leq 2N - 3$.*
- (ii) $\dim M(f)_{2N-3} - \tau = \sum_{j=1}^{r} g_j$ *if and only if $H^2(U)$ satisfies $F^2H^2(U) = P^2H^2(U)$.*

(B) *$max(r - 1 + t - \sum_{j=1}^{r} g_j, r - 1) \leq \dim ER(f)_{N-2} \leq r - 1 + t$. In particular, $\dim ER(f)_{N-2} = r - 1 + t$ if $g_j = 0$ for all j.*

Example

Let $C : f = (x^3 + y^3 + z^3)^3 + (x^3 + 2y^3 + 3z^3)^3 = 0$. C is the union of 3 smooth curves, and have 9 triple points as singularities. Using Singular we can find $\dim M(f)_{16} = \tau = 36$. Hence, one has a strict inequality in (A)

$$\dim M(f)_{16} - \tau = 0 < 3 = \sum_{j=1}^{3} g_j.$$

Moreover, the inequalities in (B) in this case are

$$8 \leq 8 \leq 9 + 2.$$

Example

Consider the line arrangements:
Pappus configuration $\mathcal{A}_1 : f = 0$

$$xyz(x-y)(y-z)(x-y-z)(2x+y+z)(2x+y-z)(-2x+5y-z) = 0,$$

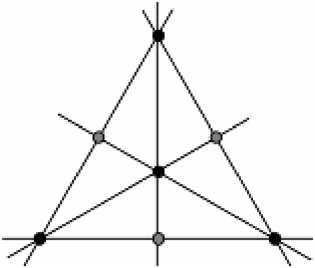
and $\mathcal{A}_2 : g = 0$

$$xyz(x+y)(x+3z)(y+z)(x+2y+z)(x+2y+3z)(4x+6y+6z) = 0.$$

Both arrangements have $N = n = t = 9$,
$P(M(f))(t) - P(M(g))(t) = t^{12} \neq 0$, and $ct(V(f)) = 11$ and $ct(V(g)) = 12$.

Example

Consider the curve $C : f = (x^2 - y^2)(y^2 - z^2)(x^2 - z^2) = 0$. C is the union of 6 lines, i.e $g_i = 0$ for $i = 1, \cdots, 6$. Hence,
$\dim M(f)_9 = 4(4) + 3 = 19 = \tau(C)$, and
$\dim ER(f)_4 = 6 - 1 + 4 = 9$.



Pure Hodge Structures

Definition

A (pure) *Hodge structure of weight m* on a finite dimensional $\mathbb{Q}$-vector space H consists of a decomposition of $H_{\mathbb{C}} = H \otimes \mathbb{C}$ into a direct sum of complex subspaces $H^{p,q}$, such that:

(i) $H_{\mathbb{C}} = \bigoplus_{p+q=m} H^{p,q}$

(ii) $\overline{H^{p,q}} = H^{q,p}$

There exists a filtration on $H_{\mathbb{C}}$, called the *Hodge Filtration*, given by

$$F^p H_{\mathbb{C}} = \bigoplus_{s \geq p} H^{s,m-s}.$$

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Mixed Hodge Structures

Definition

A *mixed Hodge structure* (MHS) is a triplet (H, W, F) where:

(i) H is a finite dimensional $\mathbb{Q}$-vector space;

(ii) W is a finite increasing filtration called the *weight filtration*

$$0 \subset W_s H \subset W_{s+1} H \subset \cdots \subset W_t H = H$$

(iii) F is a finite decreasing filtration on $H_{\mathbb{C}}$ called the *Hodge filtration*

$$H \supset F^p H \supset F^{p+1} H \supset \cdots \supset F^q H \supset 0$$

such that $(Gr_k^W H, F)$ is a Hodge structure of weight k for all k.

The induced filtration is given by

$$F^p(Gr_k^W H)_{\mathbb{C}} = (F^p H_{\mathbb{C}} \cap W_k H_{\mathbb{C}} + W_{k-1} H_{\mathbb{C}})/W_{k-1} H_{\mathbb{C}}.$$

When (H, W, F) is a MHS we can define the *mixed Hodge numbers* by

$$h^{p,q}(H) = \dim\ Gr_F^p Gr_{p+q}^W H_{\mathbb{C}}.$$

Theorem (Deligne 1971)

Let X be a quasi-projective variety, then $H^*(X,\mathbb{Q})$ has a MHS, such that for all $m \geq 0$,

- *The weight filtration W on $H^m(X,\mathbb{Q})$ satisfies*

$$0 = W_{-1} \subset W_0 \subset \cdots \subset W_{2m} = H^m(X;\mathbb{Q});$$

  for $m \geq n = \dim X$, we also have $W_{2n} = \cdots = W_{2m}$;
- *The Hodge filtration F on $H^m(X;\mathbb{C})$ satisfies $H^m(X;\mathbb{C}) = F^0 \supset \cdots \supset F^{m+1} = 0$. For $n = \dim X$, we also have $F^{n+1} = 0$.*

Theorem (Deligne, 1971)

- *If X is a smooth variety, then $W_{m-1}H^m(X,\mathbb{Q}) = 0$ (i.e., all weights on $H^m(X;\mathbb{Q})$ are $\geq m$) and $W_mH^m(X,\mathbb{Q}) = j^*H^m(\overline{X},\mathbb{Q})$ for any compactification $j : X \hookrightarrow \overline{X}$;*
- *If X is a projective variety, then $W_mH^m(X,\mathbb{Q}) = H^m(X,\mathbb{Q})$ (i.e., all weights on $H^m(X;\mathbb{Q})$ are $\leq m$) and $W_{m-1} = \ker p^*$ for any proper map $p : \tilde{X} \to X$ with $\tilde{X}$ smooth.*

Example

If X is a smooth projective variety, then the cohomology group $H^m(X,\mathbb{Q})$ has a pure Hodge structure of weight m, for all $m \geq 0$.

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Hodge Theory of Plane Curve Complement

Let $C \subset \mathbb{P}^2$ be a curve defined by $f = 0$ for $f \in S_N$, and $U = \mathbb{P}^2 \setminus C$.

In particular, for $m = 2$, the Hodge filtration is given by:

$$H^2(U) = F^0 = F^1 \supset F^2 \supset F^3 = 0$$

Theorem

Let $C \subset \mathbb{P}^2$ be a curve of degree N, and $U = \mathbb{P}^2 \setminus C$. Suppose that C has only n nodes and t triple points. Set $g_j = g(C_j)$, where the $\{C_j\}_j$ are the irreducible components of C whose number is r. Then one has

$$\dim Gr_F^1 H^2(U, \mathbb{C}) = \sum_{j=1}^{r} g_j$$

and

$$\dim Gr_F^2 H^2(U, \mathbb{C}) = \frac{(N-1)(N-2)}{2} - t.$$

Remark

The weight filtration on $H^2(U)$ is:

$$0 \subset W_3 \subset W_4 = H^2(U).$$

Corollary

(i) $h^{2,1}(H^2(U)) = h^{1,2}(H^2(U)) = \sum_{j=1}^r g_j$.

(ii) $h^{2,2}(H^2(U)) = \frac{(N-1)(N-2)}{2} - \sum_{j=1}^r g_j - t$.

(iii) $b_2(U) = \frac{(N-1)(N-2)}{2} + \sum_{j=1}^r g_j - t$, *where $b_2(U)$ denotes the second Betti number of the complement U.*

In particular, it follows that $H^2(U)$ is pure of type $(2,2)$ when $g_j = 0$ for all j, a well known property in the case of line arrangements.

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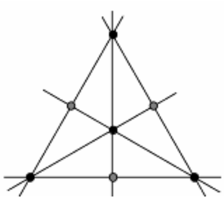
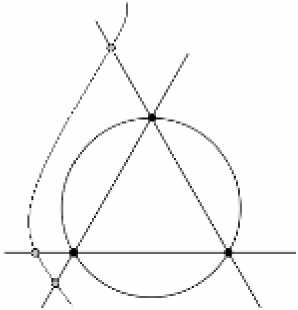
Example

$g_i = 0$ for every $i = 1, \cdots 6$, $N = 6$, and $t = 4$. Then, $\dim H^1 = 5$, and we get $\dim Gr_F^1 H^2(U, \mathbb{C}) = \dim \frac{F^1}{F^2} = 0$ and $\dim Gr_F^2 H^2(U, \mathbb{C}) = \dim F^2 = 6$. Hence $b_2(U) = 6$.



Example

$C:\ xyz(x^2y+x^2z+y^2x+y^2z+z^2x+z^2y)=0$.It has 3 triple points and 3 nodes. We have $g_1=g_2=g_3=0$, $g_4=1$, $N=6$. Then $\dim H^1(U)=3$, $\dim Gr_F^1H^2(U,\mathbb{C})=1$ and $\dim Gr_F^2H^2(U,\mathbb{C})=7$, and $b_2(U)=1+7=8$.



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Generalization

Theorem

$C \subset \mathbb{P}^2$ with isolated singularities, then

$$\dim Gr_F^1 H^2(U, \mathbb{C}) = \sum_{j=1}^{r} g_j$$

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Let $C \subset \mathbb{P}^2$ be a curve of degree N having only ordinary singular points of multiplicity at most 4. If $U = \mathbb{P}^2 \setminus C$, then one has

$$\dim Gr_F^2 H^2(U, \mathbb{C}) = \frac{(N-1)(N-2)}{2} - t - 3s + b_4^2.$$

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Thank you for your attention!