Conformal Clifford Algebras and Image Viewpoints Orbit

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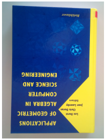

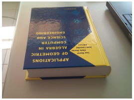

Some viewpoints of a planar object.

- Construct powerful modelings of image viewpoints and viewpoint changes
- Use these modelings to linearize the viewpoint changes by encoding them through a group of linear transformations

- **Projective geometry** : not linear $\Longrightarrow$ Complication of the calculations and algorithms (see [1]).
- Consider particular viewpoint changes : dilations for instance (see [2]).
- Use a more powerful framework for modelings : **Conformal Clifford algebras**.

[1] R. Hartley and A. Zisserman. Multiple View Geometry in Computer Vision. Cambridge University Press, ISBN : 0521540518, 2004.

[2] D-G. Lowe. Distinctive Image Features from Scale-Invariant Keypoints. International Journal of Computer Vision, 2, pages 91-110, 2004.

1. Projective modelings : preliminary calculations
2. Clifford algebras and conformal models of $\mathbb{R}^2$ in the Minkowski space :
   - Definitions and examples
   - Horospheres
3. Conformal modelings : two approaches
   - A conformal image : mapping defined on a horosphere
   - A conformal viewpoint change : horospheres change
   - The set of all the viewpoint changes :
     1. approach 1 $\Longrightarrow$ group
     2. approach 2 $\Longrightarrow$ groupoïd
   - Approach 1 *versus* approach 2

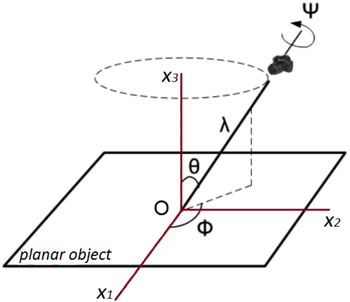

FIGURE: The extrinsic parameters $(\theta, \phi, \psi, \lambda, t_1, t_2)$ of the camera (where $t_1 = t_2 = 0$)

Frontal viewpoint $\Longleftrightarrow \theta = 0$

G. Yu and J-M. Morel. ASIFT : An Algorithm for Fully Affine Invariant Comparison Image Processing On Line, 2011, 1

A projective image is a mapping :

$$I_\theta : h_\theta(\mathbb{R}^2) \subset \mathbb{P}^2\mathbb{R} \longrightarrow \mathbb{R}. \tag{1}$$

where h_θ is a homography of the real projective plane $\mathbb{P}^2\mathbb{R}$ called perspective distortion that satisfies :

$$h_\theta[\pi(x_1,\ x_2,\ 1)] = \pi[M_\theta.(x_1,\ x_2,\ 1)], \tag{2}$$

and

$$M_\theta = \begin{pmatrix} cos\theta & 0 & 0 \\ 0 & 1 & 0 \\ -sin\theta & 0 & 1 \end{pmatrix}. \tag{3}$$

$\theta \in [0, \pi/2[$ is the latitude angle of the camera and $h_\theta(\mathbb{R}^2)$ is a perspective plane and $\pi : \mathbb{R}^3 - \{0\} \longrightarrow \mathbb{P}_2\mathbb{R}$ is the canonical projection.

to conformal

Projective pinhole camera

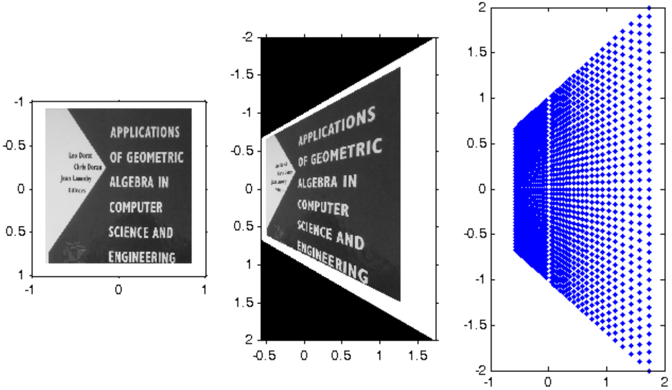

FIGURE: A frontal image of a planar object (left), a slanted view of the object for $\theta = \pi/6$ (middle) and a representation of the spacial domain of the distorted image (right).

We denote by $\mathbb{R}^{p,q}$ the space $\mathbb{R}^n$ equipped with the quadratic form Q of signature (p, q) with $p + q = n$.

The Clifford algebra (or geometric algebra) $\mathbb{R}_{p,q}$ of $\mathbb{R}^{p,q}$ is an associative algebra that contains the vectors of $\mathbb{R}^{p,q}$ and the scalars of $\mathbb{R}$ (see [1]).

The Clifford multiplication that defines $\mathbb{R}_{p,q}$ as a unitary ring is called geometric product.

If $\{e_1, e_2, ...e_n\}$ is a basis of $\mathbb{R}^{p,q}$, then $\mathbb{R}_{p,q}$ is the algebra generated by the vectors e_i. It is of dimension 2^n and admits the set

$$\{1, e_{i_1} ... e_{i_k},\ i_1 < ... < i_k,\ k \in \{1, ..., n\}\} \tag{4}$$

as basis.

D. Hestenes. New Foundations for Classical Mechanics (Fundamental Theories of Physics) Springer ; 2nd edition, 1999.

Examples :

- $\mathbb{R}_{0,1}$ is isomorphic to the commutative field of complex numbers
- $\mathbb{R}_{0,2}$ is isomorphic to the non-commutative field of quaternions

The Spin group

It is the set of the following elements of $\mathbb{R}_{p,q}$

$$Spin(p,q) \quad := \quad \{\sigma = \Pi_{i=1}^{2k} u_i,\ |Q(u_i)| = 1\} \tag{5}$$

Consider the isotropic basis $\{e_{\infty,\alpha}, e_{0,\alpha}\}$ of $\mathbb{R}^{1,1}$ (for $\alpha > 0$) :

$$e_{\infty,\alpha} = (\alpha, \alpha) \quad \text{and} \quad e_{0,\alpha} = \frac{1}{2}(-\frac{1}{\alpha}, \frac{1}{\alpha}) \tag{6}$$

satisfying $e_{\infty,\alpha} \cdot e_{0,\alpha} = -1$.

The space $\mathbb{R}^{3,1}$ is decomposed into a conformal split

$$\mathbb{R}^{3,1} = \mathbb{R}^2 \oplus \mathbb{R}^{1,1} \tag{7}$$

where $\{e_1, e_2\}$ is an orthonormal basis of $\mathbb{R}^2$. Thus, $\{e_1, e_2, e_{\infty,\alpha}, e_{0,\alpha}\}$ is a basis of $\mathbb{R}^{3,1}$.

H. Li, D. Hestenes and A. Rockwood. Sommer, G. (Ed.). Generalized homogeneous coordinates for computational geometry. Geometric computing with Clifford algebra, Springer-Verlag, 2001, 27-59.

to approach2

The horosphere H_α associated with the basis $\{e_{\infty,\alpha}, e_{0,\alpha}\}$ is the set of normalized isotropic vectors of $\mathbb{R}^{3,1}$:

$$H_\alpha = \{X \in \mathbb{R}^{3,1};\ X^2 = 0 \textit{ and } X \cdot e_{\infty,\alpha} = -1\}. \tag{8}$$

More precisely, $H_\alpha = \varphi_\alpha(\mathbb{R}^2)$ where φ_α is a global parametrization

$$\varphi_\alpha : \mathbb{R}^2 \longrightarrow H_\alpha \tag{9}$$

that sends $x \in \mathbb{R}^2$ to

$$X_\alpha = \varphi_\alpha(x) = x + \frac{1}{2}x^2 e_{\infty,\alpha} + e_{0,\alpha}. \tag{10}$$

to projective

Proposition

An image is the data of a one parameter horosphere H_{α_θ} encoding the latitude angle θ and a mapping $\bar{I}_{\alpha_\theta}$ defined on H_{α_θ} by

$$\begin{aligned} \bar{I}_{\alpha_\theta} : \{H_{\alpha_\theta}, \varphi_{\alpha_\theta}\} &\longrightarrow \mathbb{R} \\ X_{\alpha_\theta} = \varphi_{\alpha_\theta}(x) &\longmapsto I_\theta \circ h_\theta \circ \varphi_{\alpha_\theta}^{-1}(X_{\alpha_\theta}) \\ &= I_\theta \circ h_\theta(x), \end{aligned} \tag{11}$$

where I_θ is the corresponding projective image.

Conformal modelings : approach 1

Conformal viewpoint change

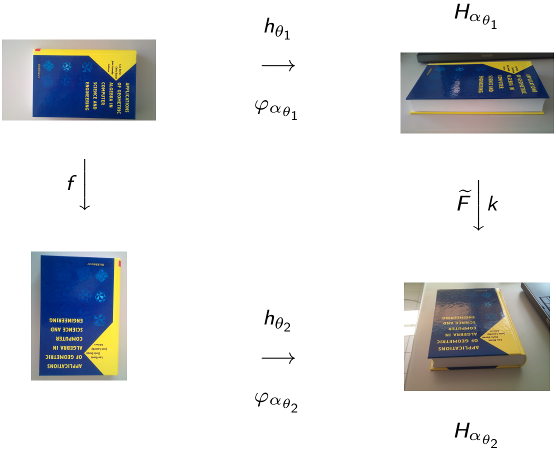

FIGURE: First line : an image I_{θ_1} (right) and its associated frontal image I_0^1 (left).
Second line : an image I_{θ_2} (right) and its associated frontal image I_0^2 (left).

The group that models the viewpoint changes is the sub-group $C_0(3,1)$ of the group $C(3,1)$ of the linear conformal transformations of $\mathbb{R}^{3,1}$ satisfying :

$$
\begin{aligned}
& \{F_{|\{H_{\alpha_\theta},\varphi_{\alpha_\theta}\}}, F \in C_0(3,1) \text{ et } \alpha_\theta > 0\} && (12)\\
= \quad & \{\widetilde{F} = \varphi_{\alpha_{\theta_2}} \circ f \circ \varphi^{-1}_{\alpha_{\theta_1}}, f \text{ similitude}, \ \alpha_{\theta_i} > 0, \}
\end{aligned}
$$

Technical calculations gives :

$$F \in C_0(3,1) \Longleftrightarrow F = F_{\overline{\alpha}} \circ F_\sigma \qquad (13)$$

where $\overline{\alpha} > 0$ and σ is a spinor encoding the similarities and

$$F_{\overline{\alpha}} \circ \varphi_{\alpha_\theta} = \varphi_{\overline{\alpha}\alpha_\theta} \qquad (14)$$

$$F_\sigma \circ \varphi_{\alpha_\theta} = \varphi_{\alpha_\theta} \circ f_{\alpha_\theta} \qquad (15)$$

Aim : Introduce more natural modelings of the latitude and the zoom of the camera.

Let $e_\infty = (1,1)$ and $e_0 = 1/2(-1,1)$ be the two isotropic vectors of the standard conformal model of $\mathbb{R}^2$ ($\alpha = 1$).

▸ to isotropic vectors

Main idea

The effect of the latitude $\theta \iff$ applying on e_∞ a rotation ρ_θ of angle θ.
The effect of the zoom $\lambda > 0 \iff$ applying on e_∞ a dilatation of ratio λ.

We propose then to introduce the vectors $\lambda e_{\infty,\theta}$ and $\lambda^{-1} e_{0,\theta}$ where :

$$e_{\infty,\theta} = \rho_\theta(e_\infty) = \begin{pmatrix} \cos\theta - \sin\theta \\ \sin\theta + \cos\theta \end{pmatrix} \qquad e_{0,\theta} = \rho_\theta(e_0) = -\frac{1}{2}\begin{pmatrix} \cos\theta + \sin\theta \\ \sin\theta - \cos\theta \end{pmatrix} \tag{16}$$

isotropic for the new metric

$$G_\theta = \rho_\theta \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \rho_\theta^{-1} = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix} \tag{17}$$

and satisfying $e_{\infty,\theta} \cdot e_{0,\theta} = -1$.

For every θ, the space is then $\mathbb{R}^4_\theta = (\mathbb{R}^4, Q_\theta)$ where

$$Q_\theta = id \oplus G_\theta \tag{18}$$

Definition

A generalized conformal representation of the Euclidean plane is the data of a two-parameter horosphere :

$$H(\lambda, \theta) = \{X \in \mathbb{R}^4_\theta,\ X^2 = 0,\ X \cdot \lambda e_{\infty,\theta} = -1\} \tag{19}$$

associated with the embedding $\varphi_{\lambda,\theta}$ of $\mathbb{R}^2$ into R^4_θ :

$$\varphi_{\lambda,\theta}(x) = x + \frac{1}{2}x^2\lambda e_{\infty,\theta} + \lambda^{-1}e_{0,\theta} \tag{20}$$

Proposition

An image is the data of a generalized conformal model $(H(\lambda,\theta),\varphi_{\lambda,\theta})$ and a mapping :

$$\begin{aligned} \bar{I}_{\lambda,\theta} : (H(\lambda,\theta),\varphi_{\lambda,\theta}) &\longrightarrow \mathbb{R} \\ X_{\lambda,\theta} = \varphi_{\lambda,\theta}(x) &\longmapsto I_{\lambda,\theta}\circ h_\theta \circ D_{\lambda^{-1}} \circ \varphi^{-1}_{\lambda,\theta}(X_{\lambda,\theta}) \\ &= I_{\lambda,\theta}\circ h_\theta \circ D_{\lambda^{-1}}(x), \end{aligned} \tag{21}$$

where $I_{\lambda,\theta}$ is the corresponding projective image.

Groupoïd :

- it is a category whose morphisms are isomorphisms
  $\Longrightarrow$ the set of morphisms of a groupoïd generalize the notion of group.

- generalization of the notion of group action :

  Consider a group G acting on a set X by

$$G \times X \longrightarrow X \tag{22}$$

  To this action corresponds a groupoïd :
  - the objects are the elements of X
  - the morphisms from x to y are the elements of G that send x to y

The groupoïd $\mathcal{PV}$ of the viewpoints and viewpoint changes :

- The objects are $(H(\lambda,\theta), \varphi_{\lambda,\theta}) \Longleftrightarrow$ representing the viewpoints
- The morphisms encode the viewpoint changes and are the compositions of the basic diagrams :
  1. (σ_t, T_t) where $\sigma_t = 1 - \frac{1}{2} t\lambda e_{\infty,\theta}$
  2. $(\sigma_\gamma, R_\gamma)$ where $\sigma_\gamma = \exp[-\frac{\gamma}{2} e_1 \wedge e_2]$
  3. (σ_δ, id) where $\sigma_\delta = \exp[-\frac{1}{2} E_\theta \ln \delta]$ and $E_\theta = e_{\infty,\theta} \wedge e_{0,\theta}$
  4. (ρ_φ, id) where

$$\rho_\varphi = id \oplus \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \tag{23}$$

in the canonical basis $\{e_1, e_2, e_+, e_-\}$ of $\mathbb{R}^4$.

The basic morphisms :

$$\begin{tikzcd} H(\lambda,\theta) \arrow[r, "\sigma_t"] & H(\lambda,\theta) \\ \mathbb{R}^2 \arrow[u, bend left, "\varphi_{\lambda,\theta}"] \arrow[r, "T_t"'] & \mathbb{R}^2 \arrow[u, bend right, "\varphi_{\lambda,\theta}"'] \end{tikzcd}$$

$$\begin{tikzcd} H(\lambda,\theta) \arrow[r, "\sigma_\delta"] & H(\delta\lambda,\theta) \\ \mathbb{R}^2 \arrow[u, bend left, "\varphi_{\lambda,\theta}"] \arrow[r, "id"'] & \mathbb{R}^2 \arrow[u, bend right, "\varphi_{\delta\lambda,\theta}"'] \end{tikzcd}$$

$$\begin{tikzcd} H(\lambda,\theta) \arrow[r, "\sigma_\gamma"] & H(\lambda,\theta) \\ \mathbb{R}^2 \arrow[u, bend left, "\varphi_{\lambda,\theta}"] \arrow[r, "R_\gamma"'] & \mathbb{R}^2 \arrow[u, bend right, "\varphi_{\lambda,\theta}"'] \end{tikzcd}$$

$$\begin{tikzcd} H(\lambda,\theta) \arrow[r, "\rho_\varphi"] & H(\lambda,\theta+\varphi) \\ \mathbb{R}^2 \arrow[u, bend left, "\varphi_{\lambda,\theta}"] \arrow[r, "id"'] & \mathbb{R}^2 \arrow[u, bend right, "\varphi_{\lambda,\theta+\varphi}"'] \end{tikzcd}$$

Approach 1 *versus* approach 2

	THE GROUP $C_0(3,1)$	THE GROUPOÏD $\mathcal{PV}$
Domain of θ	$]0, \frac{\pi}{2}[$ $\theta \neq 0$	S^1 all values
Modeling of θ	bijection $\theta \mapsto \alpha_\theta$ $\alpha_\theta = \cot\theta$	more natural action of θ = rotation of e_∞
Modeling of the dilations of ratio δ	transformations of $\mathbb{R}^2$ $x \longmapsto \delta x$	zoom change $\lambda e_\infty \longmapsto \delta\lambda e_\infty$
Modeling of the viewpoint changes	sub-group of $C(3,1)$	morphisms of groupoïd $\mathcal{PV}$

- Modelings in the projective geometry
- Conformal model of $\mathbb{R}^2$ in the Minkowski space
- Conformal modelings in computer vision : approach 1
  1. constant metric (Minkowski)
  2. embedding the domain of the projective image in $\mathbb{R}^{3,1}$
  3. an image is defined on a one-parameter horosphere H_{α_θ}
  4. construction of a group of linear conformal transformations $\mathbb{R}^{3,1}$ encoding the viewpoint changes
- Conformal modelings in computer vision : approche 2
  1. generalized conformal model : metric change for every θ
  2. an image is defined on a two-parameter horosphere $H_{\lambda,\theta}$
  3. construction of a groupoïd whose morphisms encode the viewpoint changes

- Calculations of viewpoint invariants by the action of the group (or the groupoïd') of the viewpoint changes on the set of conformal images.
- Practical implementation of algorithms in computer vision for the search of viewpoint invariants.

THANK YOU FOR YOUR ATTENTION!

Covariant detector by the action of $C_0(3,1)$ on $\bar{\mathbb{I}}$

It is a functional $\Psi : \bar{\mathbb{I}} \times C_0(3,1) \longrightarrow \mathbb{R}$ differentiable according to a parametrization of $C_0(3,1)$ and satisfying

1. existance of the canonical element : for all $\bar{I}_{\alpha_\theta} \in \bar{\mathbb{I}}$, there exists $\overline{F}(\bar{I}_{\alpha_\theta}) \in C_0(3,1)$ such that
$$\nabla\Psi(\bar{I}_{\alpha_\theta}, \overline{F}(\bar{I}_{\alpha_\theta})) = 0. \tag{24}$$
This element $\overline{F}(\bar{I}_{\alpha_\theta})$ of the group is called canonical element of $\bar{I}_{\alpha_\theta}$,
2. transversality condition : the Hessian matrix of Ψ is non-degenerate *i.e*
$$\det[H_\Psi(\bar{I}_{\alpha_\theta}, \overline{F}(\bar{I}_{\alpha_\theta}))] \neq 0, \tag{25}$$
for all $(\bar{I}_{\alpha_\theta}, \overline{F}(\bar{I}_{\alpha_\theta}))$ satisfying (24),
3. covariance condition : if $\nabla\Psi(\bar{I}_{\alpha_\theta}, \overline{F}(\bar{I}_{\alpha_\theta})) = 0$ then
$$\nabla\Psi(F * \bar{I}_{\alpha_\theta}, F \circ \overline{F}(\bar{I}_{\alpha_\theta})) = 0 \tag{26}$$
for all $F \in C_0(3,1)$.

$\bar{\mathbb{I}}$ is the set of conformal images $\bar{I}_{\alpha_\theta}$ and $*$ denotes the action of $C_0(3,1)$ on $\bar{\mathbb{I}}$.

It is a functional Φ defined on $\overline{\mathbb{I}}$ by :

$$\Phi(\overline{I}_{\alpha_\theta}) = [\overline{F}(\overline{I}_{\alpha_\theta})]^{-1} * \overline{I}_{\alpha_\theta}. \tag{27}$$

It is a complete invariant by the action $*$ of $C_0(3,1)$ on $\overline{\mathbb{I}}$:

$$\Phi(\overline{I}_{\alpha_{\theta_1}}) = \Phi(\overline{I'}_{\alpha_{\theta_2}}) \Longleftrightarrow \exists F \in C_0(3,1) \text{ tel que } F * \overline{I}_{\alpha_{\theta_1}} = \overline{I'}_{\alpha_{\theta_2}}. \tag{28}$$