The horoboundary of Riemannian manifolds

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Outline

Rays, parallels, Busemann functions...

XIX c. Beltrami : Desargues spaces

- the only simply connected are $\mathbb{E}^n$, $\mathbb{S}^n$, $\mathbb{H}^n$

late XIX c. Hadamard : nonpositively curved surfaces in $\mathbb{E}^3$

- unicity of geodesics in a homotopy class
- ends (cusps, funnels)
- rays asymptotic to cusps and funnels

1920's Cartan : generalization to higher dimension (Cartan-Hadamard manifolds)

- $exp_o : T_oX \to X$ diffeomorphism
- no geodesic loops, no critical points...

1940's Busemann : Desargues *geodesic spaces*

- theory of parallels & Busemann functions

$\rightsquigarrow$ *1970's* Gromov : functional compactification of general Riemannian manifolds

Some other applications of the Busemann functions:

Soul Theorem (Cheeger-Gromoll-Meyer), *Toponogov' Splitting Theorem*, *Harmonic* and (noncompact) *Symmetric spaces*, *dynamics of Kleinian groups*...

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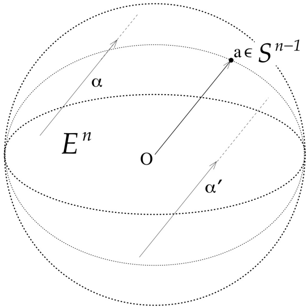
Compactifying by adding “directions”

Example. $X = \mathbb{E}^n$

$$\partial X = \{\text{half-lines}\}/_{\text{oriented parallelism}} \cong \mathbb{S}^{n-1}$$

$$\overline{X} = X \cup \partial X \cong \mathbb{B}^{n-1}$$



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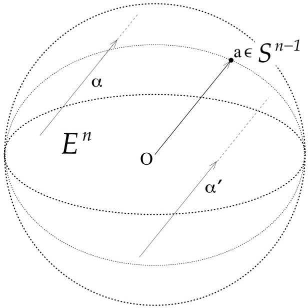

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Näif idea: X general, complete Riemannian manifold

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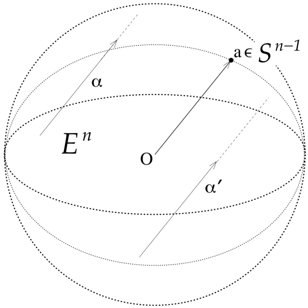

Näif idea: X general, complete Riemannian manifold

$$\mathcal{R}(X) = \{\text{rays of } X\}$$

$\alpha : \mathbb{R}^+ \to X$ is a *ray* if it is globally minimizing
i.e. $\ell(\alpha; s, t) = d(\alpha(s), \alpha(t))$ for all $s, t \geq 0$

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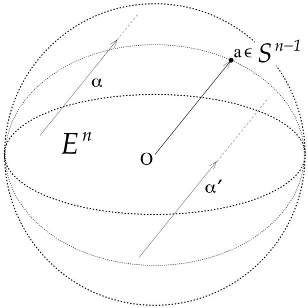

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$$\mathcal{R}(X) = \{\text{rays of } X\}$$
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(with some reasonable topology to be defined)

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oriented parallelism =?
(on general Riemannian manifolds)

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Parallelism for rays on a general, complete Riemannian manifold X:

- α *and* β *are metrically asymptotic* (i.e. $d_\infty(\alpha, \beta) < \infty$) if $\sup_{t \geq 0} d(\alpha(t), \beta(t)) < +\infty$
- α *tends visually to* β (i.e. $\alpha \succ \beta$) –β *coray to* α–

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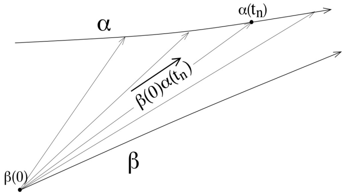
$\overrightarrow{xy}$ = *a minimizing geodesic segment from x to y*



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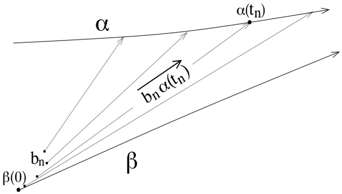

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Technical fact: the correct definition of visual convergence asks for a sequence $b_n \to \beta(0)$ with $\overrightarrow{b_n\alpha(t_n)} \to \beta$

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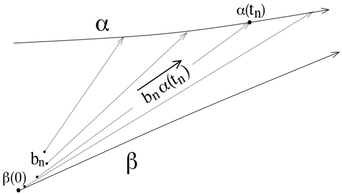

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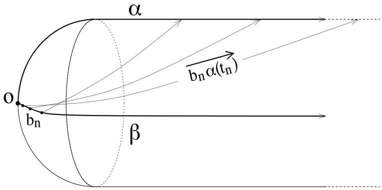
Technical fact: the correct definition of visual convergence asks for a sequence $b_n \to \beta(0)$ with $\overrightarrow{b_n\alpha(t_n)} \to \beta$

Otherwise, on a spherical-capped cylinder with pole o, different meridians would not be visually equivalent rays from o!



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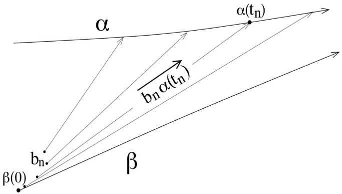
  (not symmetric, apriori)



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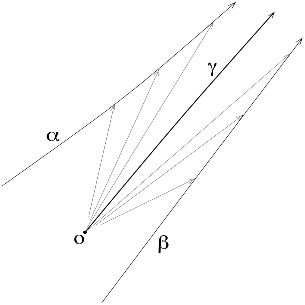

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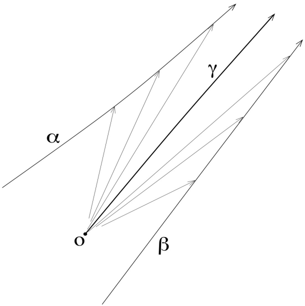

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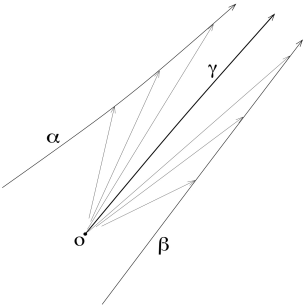

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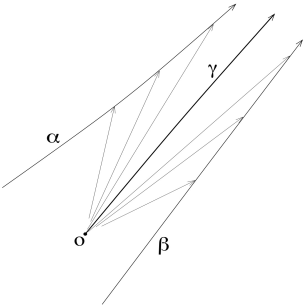

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Theorem. [Folklore: Busemann, Shihoama et al.]

Two rays α and β are $\underbrace{\text{visually asymptotic from every point } o}_{\text{"visually asymptotic"}}$ iff $B_\alpha = B_\beta$

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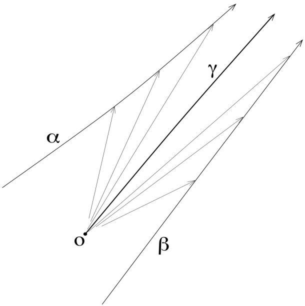

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B_α = *the Busemann function of the ray* α

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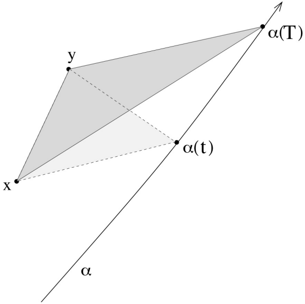
Busemann function of a ray on a general, complete Riemannian manifold X:

$$B_\alpha(x,y) = \lim_{t\to+\infty} x\alpha(t) - \alpha(t)y$$

(asymptotic defect of triangles xyp with third vertex $p = \alpha(t)$, $t \gg 0$)



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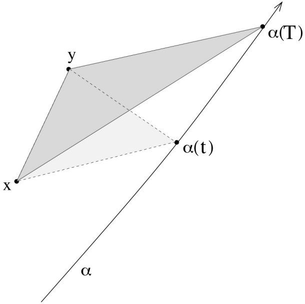

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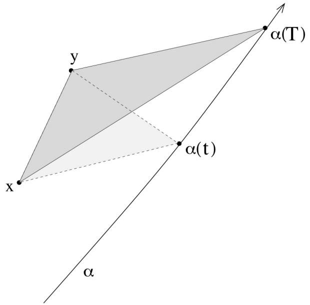

Example: $X = \mathbb{E}^n$

$\overrightarrow{xy}$ parallel to α $\Leftrightarrow xy + y\alpha(t) - x\alpha(t) \to 0$

(as x, y are fixed) $\Leftrightarrow B_\alpha(x,y) = d(x,y)$

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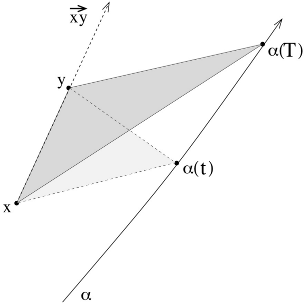

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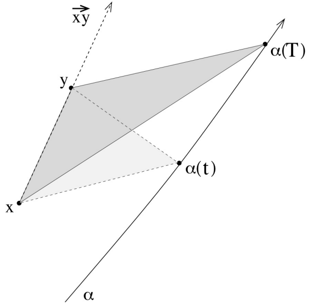

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$$\Downarrow$$

if $B_\alpha = B_\beta$ then $\alpha \succ \gamma$ iff $\beta \succ \gamma$ for every γ (and reciprocally)

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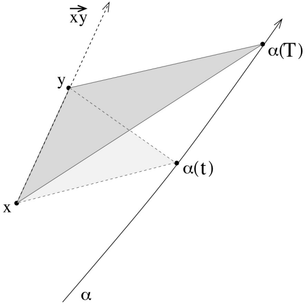

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Theorem. [Folklore: Busemann, Shihoama et al.]

Two rays α and β are visually asymptotic from every point o iff $B_\alpha = B_\beta$.

Compactifying by adding "directions"

$X =$ general, complete Riemannian manifold

$\partial X \doteq \partial_m X = \mathcal{R}(X)/_{(d_\infty(\alpha,\beta)<+\infty)}$ i.e. modulo metric asymptoticity

or

$\partial X \doteq \partial_v X = \mathcal{R}(X)/_{(B_\alpha = B_\beta)}$ i.e. modulo visual asymptoticity

$\rightsquigarrow \overline{X} = X \cup \partial X$ compactifies? what topology?

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> $\mathcal{R}(X)$ has the uniform topology (u.c. on compacts)
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Theorem. [Folklore: Eberlein, O' Neill...]

Let X be a Cartan-Hadamard manifold (= complete, simply connected with $k(X) \leq 0$).

(i) $d_\infty(\alpha,\beta) < +\infty \;\Leftrightarrow\; B_\alpha = B_\beta$, so $X(\infty) \doteq \partial_m X = \partial_v X$ = the *visual boundary*

(ii) $\overline{X}$ has a natural "visual" topology such that $X \hookrightarrow \overline{X}$ is a topological embedding
$(x_n \to \xi \in X(\infty)$ iff $\angle_o x_n, \alpha(n) \to 0 \; \exists \alpha \in \xi, \exists o \in X)$

(iii) $\overline{X}$ is a compact, metrizable space.

Compactifying by adding "directions"

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$\partial X \doteq \partial_m X = \mathcal{R}(X)/_{(d_\infty(\alpha,\beta)<+\infty)}$ i.e. modulo metric asymptoticity

or

$\partial X \doteq \partial_v X = \mathcal{R}(X)/_{(B_\alpha = B_\beta)}$ i.e. modulo visual asymptoticity

$\rightsquigarrow \overline{X} = X \cup \partial X$ compactifies? what topology?

$\mathcal{R}(X)$ has the uniform topology (u.c. on compacts) which means: $\alpha_n \to \alpha$ iff $\alpha_n'(0) \to \alpha'(0)$.

Theorem. [Folklore: Eberlein, O' Neill...]

Let X be a Cartan-Hadamard manifold (= complete, simply connected with $k(X) \leq 0$).

(i) $d_\infty(\alpha,\beta) < +\infty \Leftrightarrow B_\alpha = B_\beta$, so $X(\infty) \doteq \partial_m X = \partial_v X$ = the *visual boundary*

(ii) $\overline{X}$ has a natural "visual" topology such that $X \hookrightarrow \overline{X}$ is a topological embedding
($x_n \to \xi \in X(\infty)$ iff $\angle_o x_n, \alpha(n) \to 0$ $\exists \alpha \in \xi$, $\exists o \in X$)

(iii) $\overline{X}$ is a compact, metrizable space.

Problem. This nice picture breaks down when $\underline{\pi_1(X) \neq (1)}$ or $\underline{k(X) \not\leq 0}$.

Compactifying by adding "directions"

Examples: flutes and ladders [Dal'Bo & Peigné & S.]

There exist hyperbolic manifolds X with rays α, β and $\alpha_n \to \alpha$ in each of the following situations:

(a) $\alpha \succ \beta$ and $\beta \succ \alpha$ but $B_\alpha \neq B_\beta$

(b) $d_\infty(\alpha, \beta) < \infty$ but $B_\alpha \neq B_\beta$

(c) $d_\infty(\alpha, \beta) = \infty$ but $B_\alpha = B_\beta$

(d) $d_\infty(\alpha_n, \alpha_m) < \infty$ and $B_{\alpha_n} = B_{\alpha_m} \forall n, m$ but $d_\infty(\alpha_n, \alpha) = \infty$ and $B_{\alpha_n} \neq B_\alpha$

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$\leadsto$ $\overline{X}$ non Hausdorff,
(for any “visual” topology
on $\partial_m X$ or $\partial_v X$)

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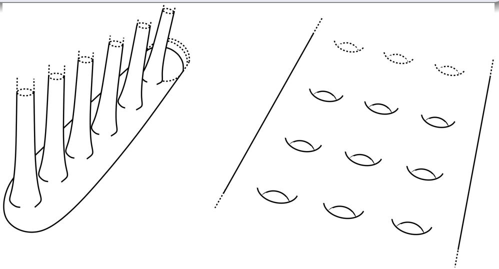

Compactifying by adding "horofunctions"

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[M.Gromov, “Hyperbolic manifolds, groups and actions” (1978)]

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$\partial_B X \doteq B(\mathcal{R}(X)) \subset \partial X$ the *Busemann points* of the boundary

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Let X be a Cartan-Hadamard n-manifold:
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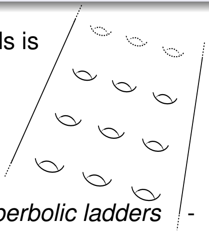

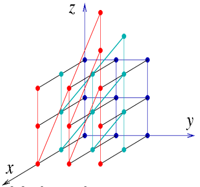

hyperbolic ladders - *Heisenberg group*

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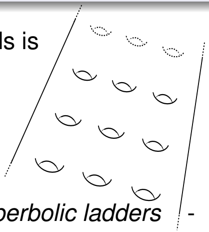
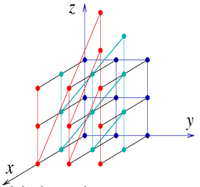
$\rightsquigarrow$ no easy "picture" of the Gromov boundary





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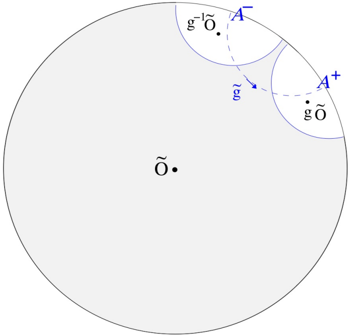

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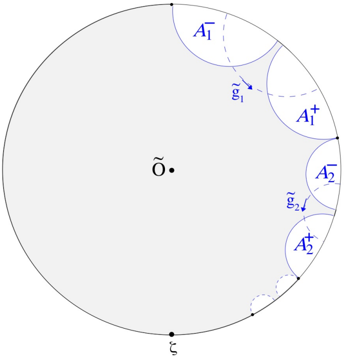

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If: • the axes $\tilde{g}_i \cap \tilde{g}_j = \emptyset$
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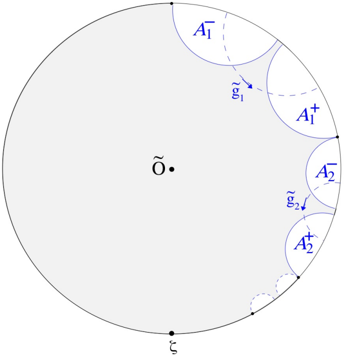

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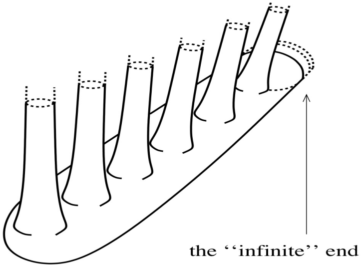
$\Rightarrow X = G \backslash \mathbb{H}^2$ hyperbolic flute

each hyperbolic $g_i \rightsquigarrow$ funnel
each hyperbolic $g_i \rightsquigarrow$ cusp
$\zeta \rightsquigarrow$ the "infinite" end e



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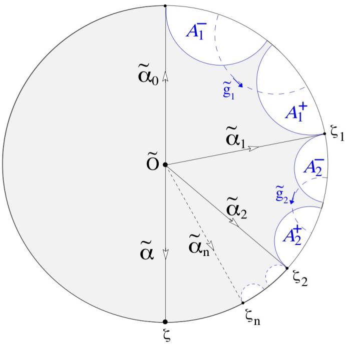
g_i hyperbolic/parabolic isometries of $\mathbb{H}^2$

Schottky : $A(g_i^{\pm 1}, \tilde{o}) \cap A(g_j^{\pm 1}, \tilde{o}) = \emptyset \quad \forall i \neq j$

For $A(g_i, \tilde{o})$ and ζ_i as in the picture: $\tilde{\alpha}_i = \overrightarrow{\tilde{o}\zeta_i}$, $\tilde{\alpha} = \overrightarrow{\tilde{o}\zeta}$

$\rightsquigarrow \alpha_i = \tilde{\alpha}_i / G$ metrically asymptotic rays on $X = G \backslash \mathbb{H}^2$
even: $d_\infty(\alpha_i, \alpha_j) = 0$, so $B_{\alpha_i} = B_{\alpha_j}$

We have $\alpha_i \to \alpha$ but: $B_\alpha \neq \lim_{i\to\infty} B_{\alpha_i} = B_{\alpha_0}$



Twisted Hyperbolic flute

Theorem. [Dal'Bo & Peigné & S.]

There exists a hyperbolic ladder $X \to \Sigma_2$ with group $G \cong \mathbb{Z}$ and rays α, α' such that:
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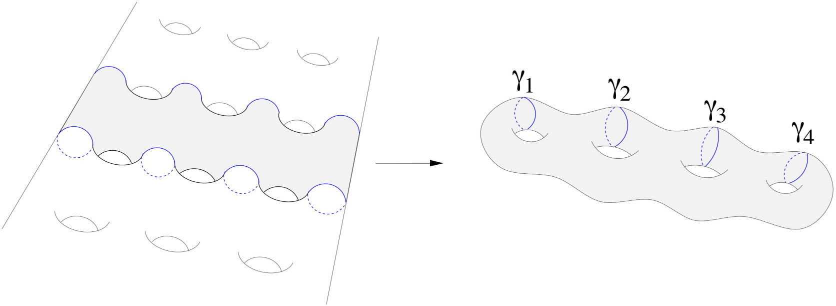
hyperbolic ladder = $\begin{cases} \mathbb{Z}\text{-covering of a closed hyperbolic surface } \Sigma_g \text{ of genus } g \geq 2 \\ \text{obtained by glueing infinitely many copies of } \Sigma_g - \bigcup_{i=1}^{g} \gamma_i \\ \quad \text{with } (\gamma_i) \text{ simple, closed non-intersecting fundamental geodesics} \end{cases}$



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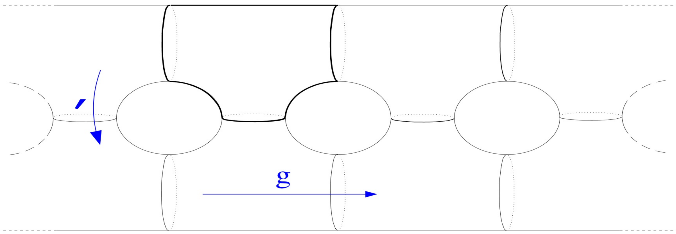

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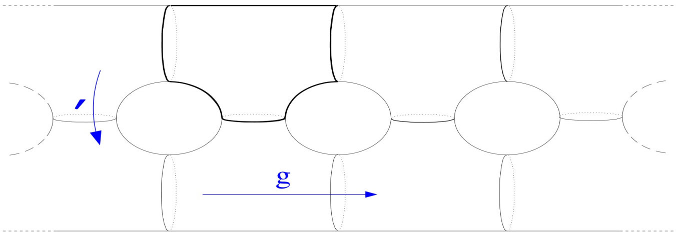

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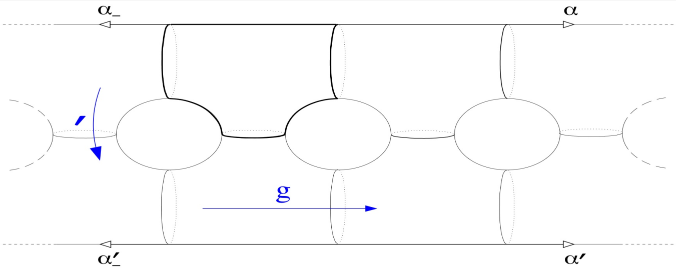

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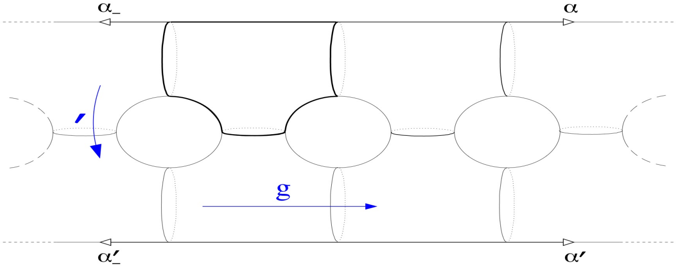

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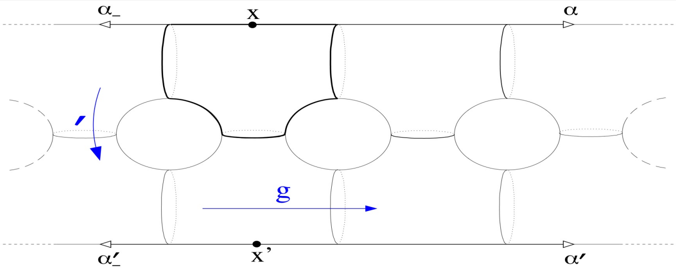

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- $B_\alpha \neq B_{\alpha'}$ as (by direct computation) $B_\alpha(x, x') > 0 \rightsquigarrow B_{\alpha'}(x, x') \stackrel{!}{=} B_\alpha(x', x) = -B_\alpha(x, x') < 0$

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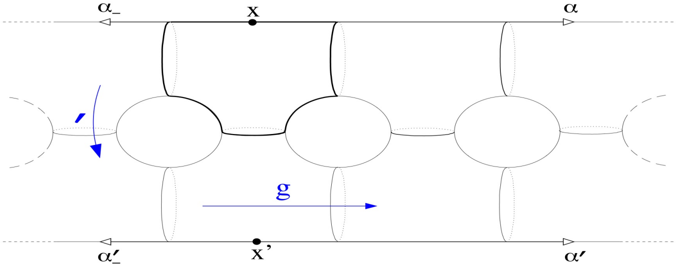

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– the hyperbolic metric $\rightsquigarrow$ every other ray is metrically *strongly* asymptotic ($d_\infty = 0$) to one of $\{\alpha, \alpha_-, \alpha', \alpha'_-\}$

$\Longrightarrow$ $\partial_B X$ has 4 points

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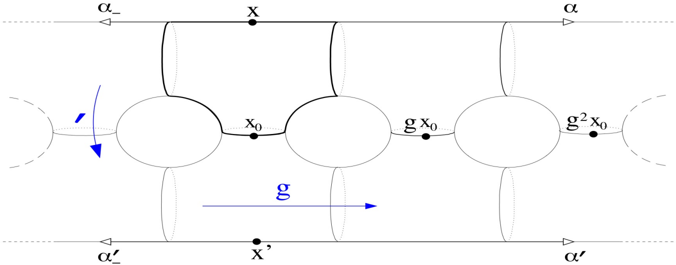

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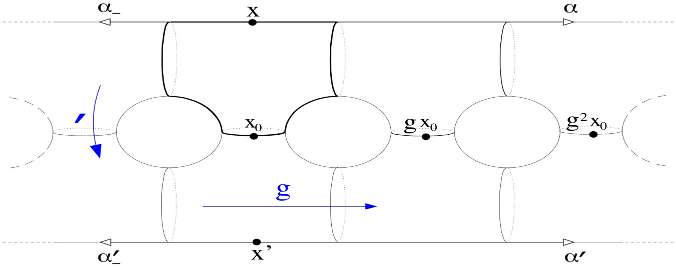

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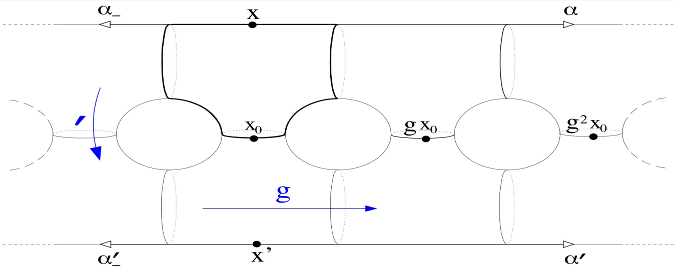

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– actually if $b_{g^i x_0} \to B_\alpha$ (let's say) $\Rightarrow$ $b_{g^i x_0} = b_{(g^i x_0)'} \to B_{\alpha'}$ so $B_\alpha = B_{\alpha'}$, contradiction.

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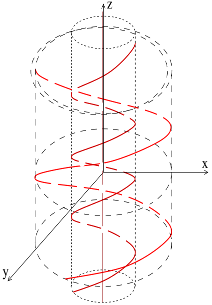

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Remark: there are no rays in (**H**, d_R) beyond *plane* geodesics.
Riemannian geodesics are:

- (minimizing) straight lines in planes parallel to $\mathbb{C}$;
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- straight vertical lines.

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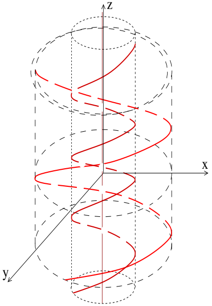

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Theorem. [Klein &Nikas (C.C. case) - Le Donne&Nicolussi&S. (Riemannian case)]

(i) The Gromov Boundary of the Heisenberg group endowed with any C.C. or left invariant Riemannian metric is homeomorphic to a 2-dimensional closed disk $\bar{D}^2$;

(ii) if d_R and d_{CC} are compatible, then they are *strongly asymptotically isometric*:

$$d_R(P, Q) - d_{CC}(P, Q) \to 0 \text{ for } d(P, Q) \to \infty$$

Ways of diverging in the Heisenberg group

4 ways of going to infinity for a sequence of points $P_n = (\vec{u}_n, z_n) \in \mathbf{H} = \mathbb{C} \times \mathbb{R}$:

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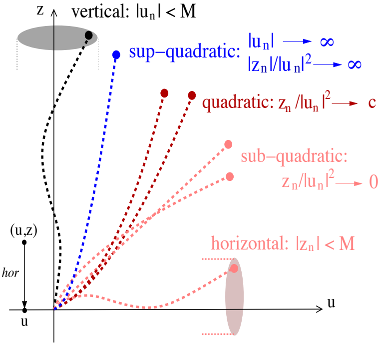

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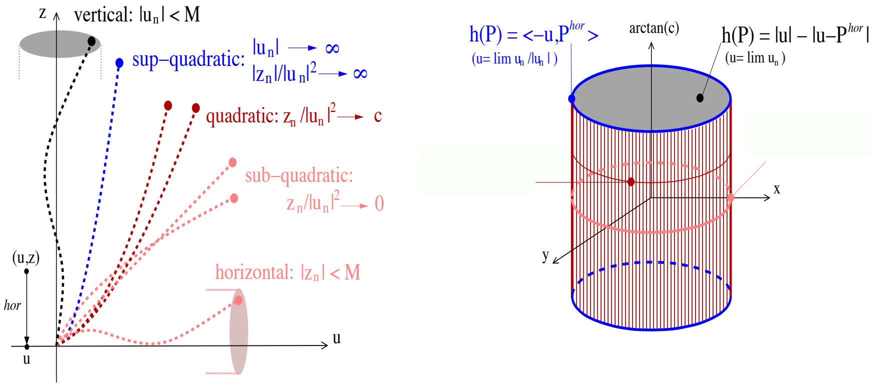

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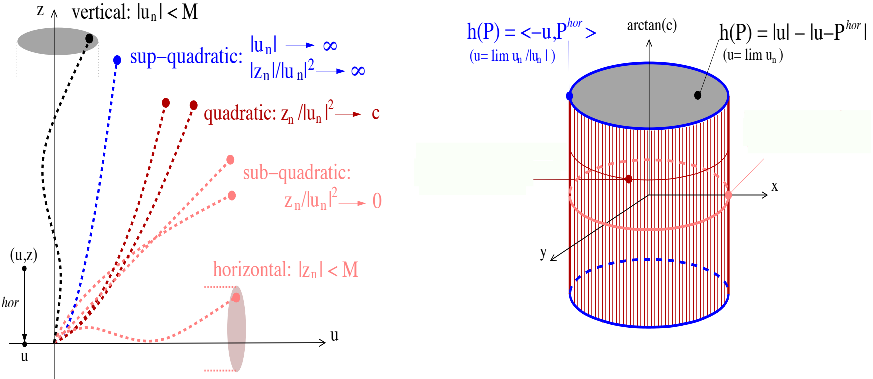

Facts: a) (correct gluing) if $|u_n| \to \infty$, then $< -\vec{u}, P >= lim_{n\to\infty} |\vec{u}_n| - |\vec{u}_n - P|$ for $\vec{u} = \lim_n \vec{u}_n / |u_n|$

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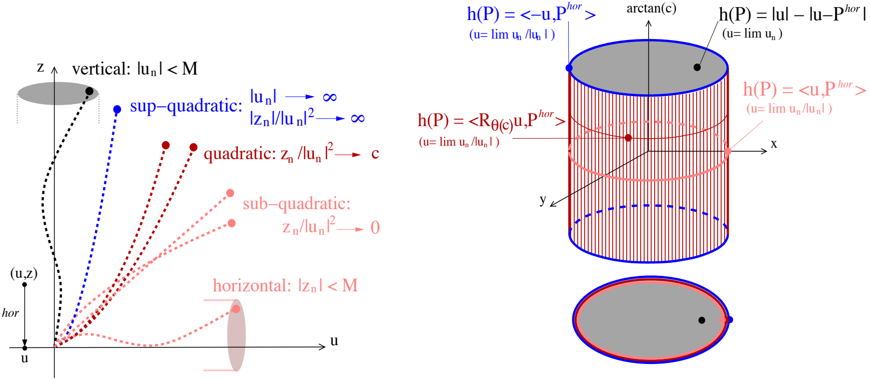

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b) (identification) if $\begin{cases} \vec{u}_n \to u \\ z_n/|\vec{u}_n|^2 \to c \end{cases}$ and $\begin{cases} \vec{u}'_n \to u \\ z'_n/|\vec{u}'_n|^2 \to c' \end{cases}$ then $h = h'$ iff $R_{\theta(c)}\vec{u} = R_{\theta(c')}\vec{u}'$

Geometrically finite manifolds

= a large class of (negatively curved) manifolds with finitely generated $\pi_1(X)$

- $dim(X) = 2 \rightsquigarrow$ same as $\pi_1(X)$ f.g.
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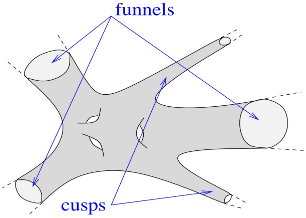

$X = G \backslash H \quad H =$ Cartan-Hadamard, $-b^2 \leq k(H) \leq -a^2 < 0$

LG the limit set of G, $\quad CG \subset H$ its convex hull

$CX = G \backslash CG \subset X$ the *Nielsen core* of X
(the smallest closed and convex subset of X containing all the geodesics which meet infinitely many often a compact set)

X is geometrically finite if some (any) ϵ-neighbourhood of CX has finite volume

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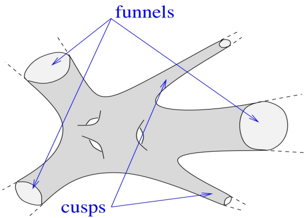

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Theorem. [Dal'Bo & Peigné & S.]

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(ii) $B : \mathcal{R}(X) \to \partial X$ is continuous and surjective

$\Longrightarrow X(\infty) \cong \mathcal{R}(X)/_{equiv.} \cong \partial X$

- if $dim(X) = 2 \rightsquigarrow \overline{X}$ is a compact surface with boundary
- if $dim(X) > 2 \rightsquigarrow \overline{X}$ is a compact manifold with boundary with a finite number of *conical singularities* (one for each conjugate class of maximal parabolic subgroups of G)

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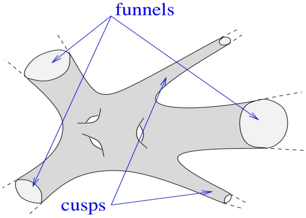

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- if $dim(X) > 2 \quad \rightsquigarrow \quad \overline{X}$ is a compact manifold with boundary with a finite number of *conical singularities* (one for each conjugate class of maximal parabolic subgroups of G)

$\xi \in \overline{X}$ is a *conical singularity* if it has a neighbourhood homeomorphic to the cone over some topological manifold)

The simplest (non-compact, non simply-connected) geometrically finite 3-manifold

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$LP = \{\xi\}$ the limit set $Ord(P) = \partial \mathbb{H}^3 - \xi$ *the discontinuity domain*

$D(P, \tilde{o}) = \{x \in \mathbb{H}^3 : d(x, \tilde{o}) \leq d(x, p^n \tilde{o}), \forall n \in \mathbb{Z}\}$ *the Dirichlet domain*

1. see $X = P \backslash D(P, \tilde{o}) = P \backslash [H_\xi \times (0, +\infty)] \cong Cil \times (0, +\infty)$
2. add the *ordinary Dirichlet points*: $X' = X \cup [\partial D(P, \tilde{o}) \cap Ord(P)] \cong Cil \times [0, +\infty)$
3. adding one point corresponding to $\xi \leftrightarrow P$ [with the topology: $x_n \to \xi$ for any diverging (x_n)]:

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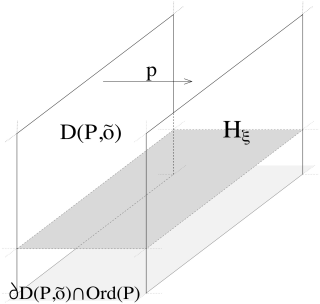

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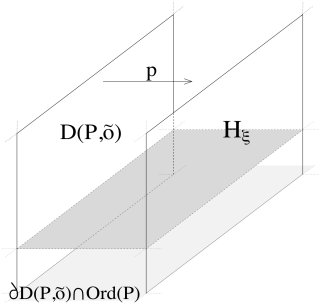

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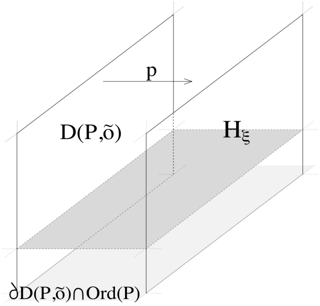

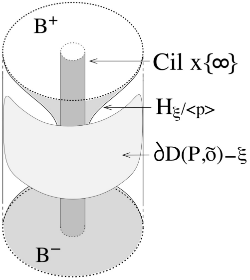

X' = solid infinite cylindrical shell

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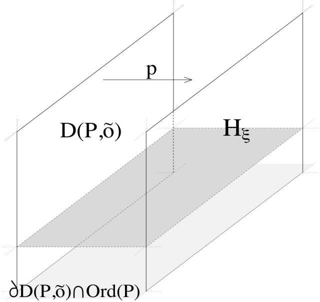

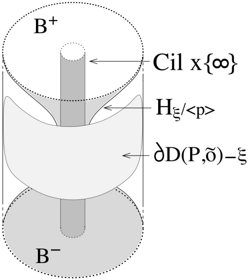

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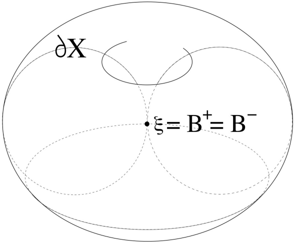

$\overline{X}$ = solid spindle torus