The effect of geometry on the eigenvalues of the Laplacian

Ahmad El Soufi
Laboratoire de Mathématiques et Physique Théorique
UMR CNRS 7350, Université François Rabelais de Tours
F-37200 Tours France

Part I : Euclidean domains

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Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and consider the eigenvalue problem

$$\begin{cases} \Delta u + \lambda u & = 0 \quad \text{in } \Omega \\ u & = 0 \quad \text{on } \ \partial\Omega. \end{cases}$$

with

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$$

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$\forall u \in C_0^\infty(\Omega)$

$$\int_\Omega u \Delta u = \int_\Omega |\nabla u|^2$$

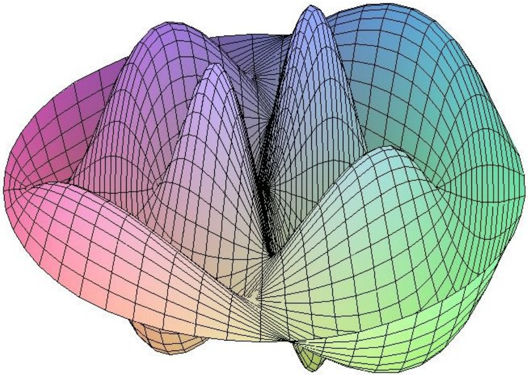
$$\lambda_1(\Omega) = \inf_{u \in C_0^\infty(\Omega)} \frac{\int_\Omega |\nabla u|^2}{\int_\Omega u^2}$$

and

$$\lambda_k(\Omega) = \inf_{E \in S_k} \sup_{u \in E} \frac{\int_\Omega |\nabla u|^2}{\int_\Omega u^2}$$

where S_k is the set of all k-dimensional vector subspaces of $C_0^\infty(\Omega)$.

Drum :



The equation that governs the vibrations of the membrane of a drum is the wave equation

$$\begin{cases} \frac{\partial^2}{\partial t^2} f - \Delta_x f & = 0 \quad \text{in } \Omega \\ f & = 0 \quad \text{on } \partial\Omega. \end{cases}$$

where $f(x,t)$ represents the height at time t above the point x.

Separation of variables :

$$f(x,t) = \omega(t)u(x)$$

satisfies the wave equation iff there exists $\lambda > 0$ such that

$$\begin{cases} \frac{\partial^2}{\partial t^2}\omega + \lambda\omega & = 0 \\ \Delta u + \lambda u & = 0 \quad \text{in } \Omega \\ u & = 0 \quad \text{on } \ \partial\Omega. \end{cases}$$

Thus

$$\omega(t) = A\cos(\sqrt{\lambda}t) + B\sin(\sqrt{\lambda}t)$$

and

$$\begin{cases} \Delta u + \lambda u & = 0 \quad \text{in } \Omega \\ u & = 0 \quad \text{on } \ \partial\Omega. \end{cases}$$

which means that $\frac{1}{2\pi}\sqrt{\lambda}$ is the frequency of the vibration.

Diffusion of heat :

$$\begin{cases} \frac{\partial}{\partial t} f(x,t) - \Delta_x f(x,t) & = 0 & \text{in } \Omega \\ f(x,0) & = g(x) & \text{in } \Omega \\ f(x,t) & = 0 & \text{on } \partial\Omega. \end{cases}$$

where $f(x,t)$ represents the temperature at the point $x \in \bar{\Omega}$ and time t. The initial data $g(x)$ is assumed to be zero on $\partial\Omega$.

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where $f(x,t)$ represents the temperature at the point $x \in \bar{\Omega}$ and time t. The initial data $g(x)$ is assumed to be zero on $\partial\Omega$. Applying the separation of variables we show that

$$f(x,t) = \sum_{n=1}^{\infty} a_k e^{-\lambda_k t} u_k(x)$$

where $\{u_k\}$ is an L^2-orthonormal basis of eigenfunctions with $\Delta u_k = \lambda_k u_k$ and $a_k = \int_\Omega g(x) u_k(x) dx$.

$$f(x,t) = e^{-\lambda_1 t}\left(a_1 u_1(x) + 0(e^{\lambda_2 - \lambda_1)t})\right) \quad \text{as } t \to \infty$$

Examples:

The rectangle $R_{a,b} = (0, a) \times (0, b)$:

$$Spec(R_{a,b}) = \left\{ \lambda_{n,m} = \pi^2 \left(\frac{n^2}{a^2} + \frac{m^2}{b^2} \right) \; : \; n \geq 1, \; m \geq 1 \right\}$$

with corresponding eigenfunctions

$$u_{n,m}(x, y) = \sin(\frac{n\pi}{a} x) \sin(\frac{m\pi}{b} y)$$

In particular, the least eigenvalue is given by

$$\lambda_1(R_{a,b}) = \pi^2 \left(\frac{1}{a^2} + \frac{1}{b^2} \right)$$

The unit disk $D_1 = \{x^2 + y^2 < 1\}$
Using polar coordinates (r, θ), we may write

$$u(r,\theta) = v_0(r) + \sum_{n\geq 1} \left(v_n(r)\cos(n\theta) + v_{-n}(r)\sin(n\theta)\right)$$

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$$r^2 v_n'' + r\ v_n' + (r^2 \lambda - n^2) v_n = 0$$

with $v_n(1) = 0$. Setting $v_n(r) = y(\sqrt{\lambda} r)$ we get, with $s = \sqrt{\lambda} r$,

$$s^2 y'' + s y' + (s^2 - n^2) y = 0$$

(Bessel equation) with $y(\sqrt{\lambda}) = 0$.
Thus, $v_n(r) = J_n(\sqrt{\lambda} r)$, where J_n is a Bessel function of the first kind. The eigenvalues of D_1 are the squares of the zeroes of J_n.
In particular,

$$\lambda_1(D_1) = j_{0,1}^2 \approx (2.4048)^2$$

where $j_{0,1}$ is the first positive zero of the Bessel function J_0.

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Weyl's asymptotic formula (1911) :

$$\lambda_k(\Omega) \approx C_n|\Omega|^{-\frac{2}{n}}k^{\frac{2}{n}}, \qquad \text{as } k \to \infty$$

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In dimension 2

$$\lambda_k(\Omega) \approx 4\pi\frac{k}{|\Omega|}, \qquad \text{as } k \to \infty$$

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Li-Yau (1983): $\forall k \geq 1$

$$\lambda_k(\Omega) \geq \frac{n}{n+2} C_n |\Omega|^{-\frac{2}{n}} k^{\frac{2}{n}}$$

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<u>Remark 1</u> : λ_k^* does not change if we add a connectedness constraint in the definition.

<u>Remark 2</u> : $\sup_{\Omega \subset \mathbb{R}^n \,;\, |\Omega|=1} \lambda_1(\Omega) = +\infty$

For example, for the rectangle $R_{a,b}$ one has

$$\lambda_1(R_{a,b}) = \pi^2 \left(\frac{1}{a^2} + \frac{1}{b^2} \right) \to \infty$$

as $a \to 0$ and $b = \frac{1}{a}$.

Faber-Krahn (1924): $\lambda_1^* = \lambda_1(B^n)\omega_n^{\frac{2}{n}} = j_{\frac{n}{2}-1,1}^2 \omega_n^{\frac{2}{n}}$,

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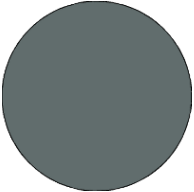

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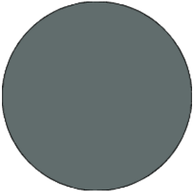
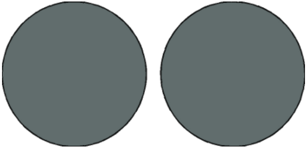


Recall that $Heat(x,t) \approx c(x)e^{-\lambda_1(\Omega)t}$

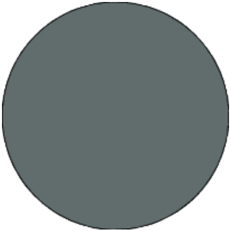
Krahn-Szegö : $\lambda_2^* = 2^{\frac{2}{n}} \lambda_1^*$

The infimum of λ_2 is achieved by the disjoint union of two balls of the same radius.



Dimension 2 :

Conjecture : $\lambda_3^* = \pi\lambda_3(disk) = \pi\ j_{1,1}^2 \approx 46.125$



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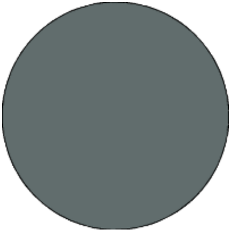
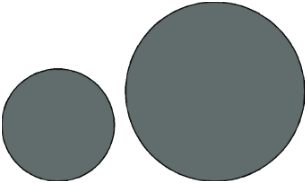


Conjecture : λ_4^* is achieved by the union of two disks whose radii are in the ratio $\frac{j_{0,1}}{j_{1,1}} \approx 1.59$ and $\lambda_4^* = \lambda_1^* + \lambda_3^* \approx 64.293$



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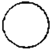
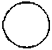
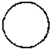
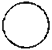
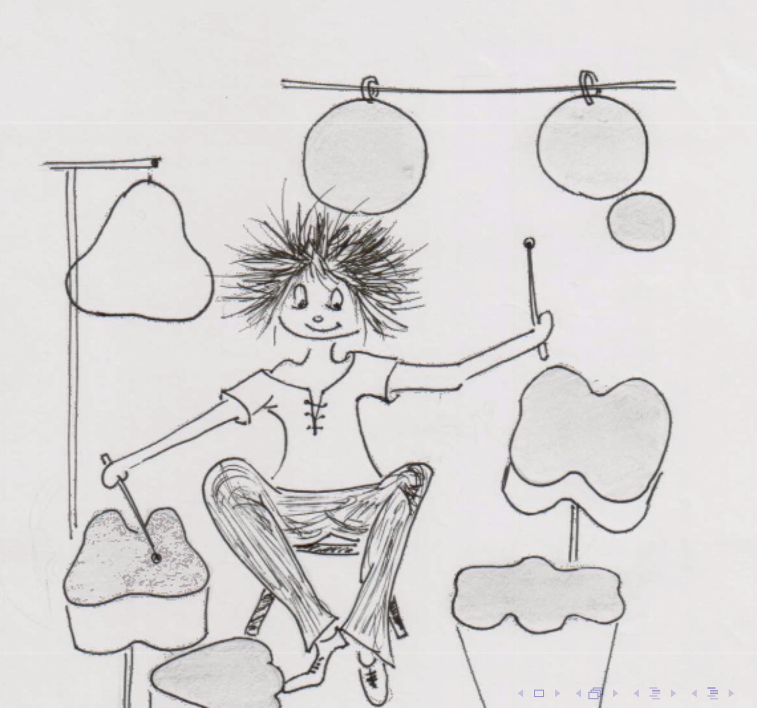
Amandine Berger (2015) : $\forall k \geq 5$, λ_k^* is not achieved by a disk or a union of disks

Edouard Oudet (2004)

No	Formes optimales de W&K		Formes optimales	
3		46.125		46.125
4		64.293		64.293
5		82.462		**78.47**
6		92.250		**88.96**
7		110.42		**107.47**
8		127.88		**119.9**
9		138.37		**133.52**
10		154.62		**143.45**



Colbois-E. (Math. Z. 2014) : The sequence ${\lambda_k^*}^{n/2}$ is subadditive, i.e., $\forall\ i_1, i_2, \cdots, i_p \in \mathbb{N}^*$ with $i_1 + i_2 + \cdots + i_p = k$,

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$${\lambda_k^*}^{n/2} \leq {\lambda_{i_1}^*}^{n/2} + {\lambda_{i_2}^*}^{n/2} + \cdots + {\lambda_{i_p}^*}^{n/2}.$$

The equality holds iff there exists a minimizing sequence Ω_N for λ_k such that each Ω_N is a disjoint union of p domains $A_1^N, \cdots, A_p^N$ with, for each $k \leq p$, A_k^N is, up to volume normalization, a minimizing sequence of domains for λ_{i_k}.

Corollary:

$$\lambda_{k+1}^{*}{}^{\frac{n}{2}} - \lambda_{k}^{*}{}^{\frac{n}{2}} \leq \lambda_{1}^{*}{}^{\frac{n}{2}} = j_{\frac{n}{2}-1,1}^{n}\omega_n$$

Corollary:

$$\lambda_{k+1}^{*\,\frac{n}{2}} - \lambda_k^{*\,\frac{n}{2}} \leq \lambda_1^{*\,\frac{n}{2}} = j_{\frac{n}{2}-1,1}^{n}\omega_n$$

In particular, in dimension 2

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Consequence : some of the numerical computations made by Oudet are not accurate. For example, he obtained $\lambda_6^* \approx 88.96$ and $\lambda_7^* \approx 107.47$, but $107.47 - 88.96$ exceeds 18.168.

Improvements of Oudet's calculations have been obtained recently by Antunes and Freitas using our theorem.

Corollary:

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Iterating the inequality of the Corollary we get

$$\lambda_k^* \leq \pi j_{0,1}^2 k$$

which, together with Polya's conjecture gives

$$4\pi k \leq \lambda_k^* \leq 5.784\ \pi k$$

Fekete's Subadditive Lemma leads to :

Corollary : The sequence $\frac{\lambda_k^*(n)}{k^{2/n}}$ converges to a positive limit with

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Thus,

$$\text{Pólya's conjecture} \quad \Leftrightarrow \quad \lim_k \frac{\lambda_k^*(n)}{k^{2/n}} = 4\pi^2 \omega_n^{-2/n}$$

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Ashbaugh-Benguria (1992):

$$\frac{\lambda_2(\Omega)}{\lambda_1(\Omega)} \leq \frac{\lambda_2(B^n)}{\lambda_1(B^n)}.$$

Yang (1991): $\forall \Omega \subset \mathbb{R}^n$ and $\forall k \geq 1$,

$$\sum_{i=1}^{k} \left(\lambda_{k+1}(\Omega) - \lambda_i(\Omega)\right)^2 \leq \frac{4}{n} \sum_{i=1}^{k} \lambda_i(\Omega) \left(\lambda_k(\Omega) - \lambda_i(\Omega)\right).$$

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Cheng-Yang (2007):

$$\frac{\lambda_{k+1}(\Omega)}{\lambda_1(\Omega)} \leq \left(1 + \frac{4}{n}\right) k^{\frac{2}{n}}.$$

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Thus, Polya's conjecture has no analogue for hypersurfaces. It is necessary to involve additional geometric quantities in order to bound the eigenvalues.

Define

$$i(M) = \sup_L \# M \cap L,$$

where L runs over the set of all lines which are transverse to M.

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If M is algebraic, then $i(M) \leq$ algebraic degree.

$i(M)$ is called "Thom's degree" of M.

Colbois-Dryden-E. (BLMS, 2009) :

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Corollary : For any convex hypersurface M, one has

$$\lambda_k(M)|M|^{\frac{2}{n}} \leq c(n)\ k^{2/n}.$$

Open problem :

$$\sup_{M \text{ convex}} \lambda_1(M)|M|^{\frac{2}{n}} = \lambda_1(\mathbb{S}^n)|\mathbb{S}^n|^{\frac{2}{n}}\ ?$$

Colbois-E.-Girouard (Crelle's 2012) :

$$\lambda_k(M)|M|^{2/n} \leq c(n)\ I(\Omega_M)^{\frac{n+2}{n}}\ k^{2/n}$$

where Ω_M is the bounded domain such that $\partial\Omega_M = M$ and

$$I(\Omega_M) = \frac{|M|}{|\Omega_M|^{\frac{n}{n+1}}}$$

is the isoperimetric ratio of Ω_M.

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Asma Hassanezhad (2011) :

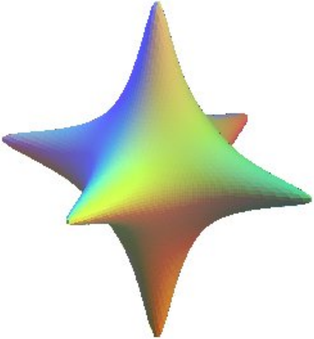
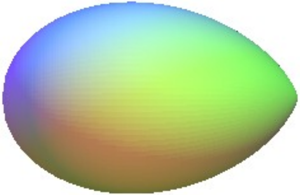
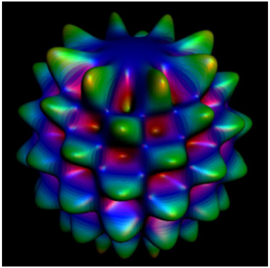
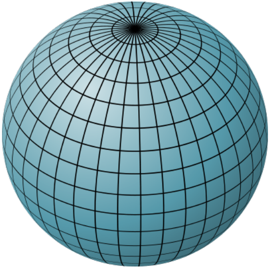
$$\sup_{|M|=1} \lambda_k(M) \leq c_1 \,\mathrm{genus}(M) + c_2 \, k$$

Here c, c_1, c_2 are universal constants.

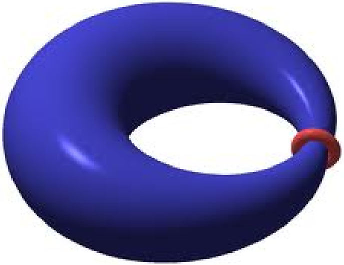
For a compact surface M_0 we define

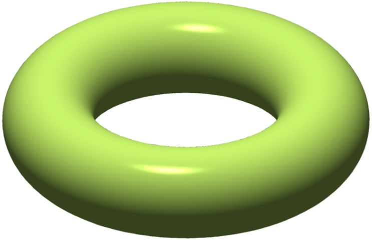
$$\lambda_k^*(M_0) = \sup\{\lambda_k(M)\,;\, M \text{ homeomorphic to } M_0 \text{ and } |M| = 1\}.$$

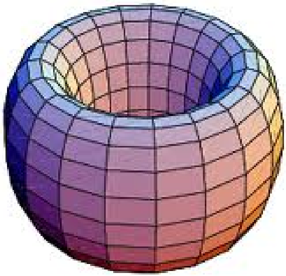
Hersch (1970): $\lambda_1^*(\mathbb{S}^2) = \lambda_1(\text{round sphere}) = 8\pi$

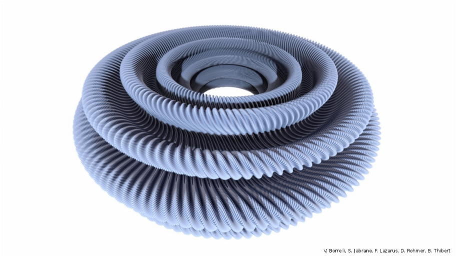





Nadirashvili (1996) : For the torus $\lambda_1^*(\mathbb{T}^2) = \frac{8\pi^2}{\sqrt{3}}$









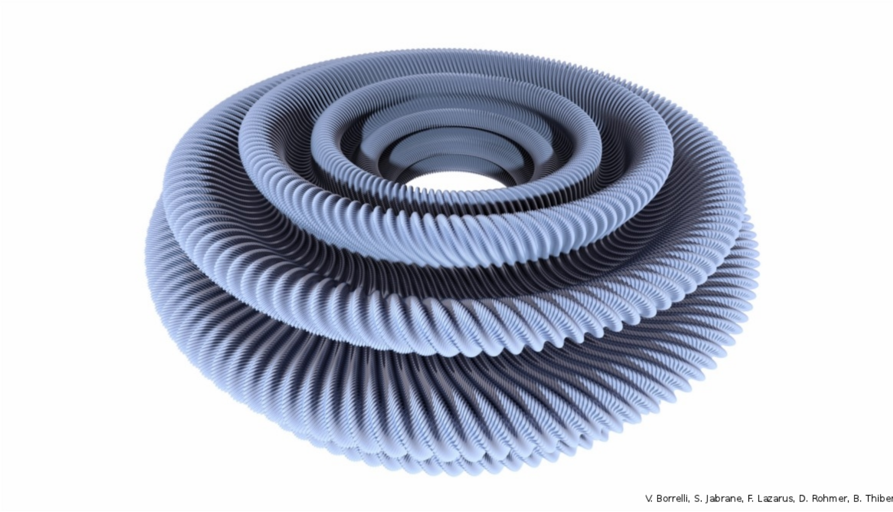


Figure : The image of a square flat torus by a C^1 isometric map (Borrelli, Jabrane, Lazarus, Thibert)

Li-Yau (1982) : For the projective plane $\lambda_1^*(\mathbb{R}P^2) = 12\pi$

E.-Giacomini-Jazar (Duke, 2006) : The Klein bottle

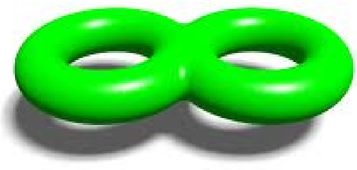
$$\lambda_1^*(\mathbb{K}^2) = 12\pi E(2\sqrt{2}/3) \simeq 13.365\,\pi,$$

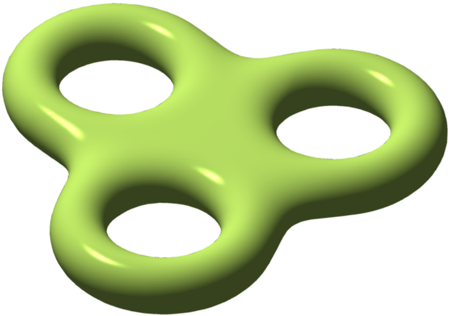
where $E(2\sqrt{2}/3)$ is the complete elliptic integral of the second kind evaluated at $\frac{2\sqrt{2}}{3}$ (based on results by Nadirashvili (1996) and Jakobson-Nadirashvili-Polterovich (2003))

Nadirashvili (2002) :

$$\lambda_2^*(\mathbb{S}^2) = 2\lambda_1^*(\mathbb{S}^2) = 16\pi.$$

Surfaces of higher genus :





$$\begin{aligned}\lambda_k^*(\gamma) &= \sup\{\lambda_k(M) \;:\; \text{M orientable, genus}(M)=\gamma \text{ and } |M|=1\} \\ &= \sup\{\lambda_k(M)|M| \;:\; \text{M orientable and genus}(M)=\gamma\}.\end{aligned}$$

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$$\lambda_k^*(\gamma) \geq \lambda_{i_1}^*(\gamma_1) + \cdots + \lambda_{i_p}^*(\gamma_p).$$

In particular,

$$\lambda_{k+1}^*(\gamma) - \lambda_k^*(\gamma) \geq 8\pi.$$

Corollary : $\forall k \geq 1$,

$$\frac{4}{5}\pi\gamma + 8\pi k - 9\pi \leq \lambda_k^*(\gamma) \leq A\gamma + Bk.$$

No PPW like inequalities since

$$\sup_M \frac{\lambda_{k+1}(M)}{\lambda_k(M)} = +\infty$$

and

$$\inf_M \frac{\lambda_{k+1}(M)}{\lambda_k(M)} = 1.$$