Beirut Lectures III: Constructions of complete quaternionic Kähler manifolds

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Some references for Lecture III

[CNS] C.–, Nardmann, Suhr (CAG, accepted), math.DG:1407.3251.

[CDL] C.–, Dyckmanns, Lindemann (PLMS '14)

[CHM] C.–, Han, Mohaupt (CMP '12).

Main idea

Use supergravity constructions to obtain new complete quaternionic Kähler manifolds

Consider the supergravity q-map, which is the composition of the supergravity r- and c-maps and its one-loop deformation.

Theorem [CHM]

The supergravity q-map associates a complete quaternionic Kähler manifold of dimension $4n + 8$ (of negative scalar curvature) with every complete projective special real manifold of dimension n.

Theorem [Dyckmanns, PhD thesis Hamburg]

The one-loop deformation $(\bar{N}, g^c_{\bar{N}})$ for $c \geq 0$ of any complete QK mf. $(\bar{N}, g_{\bar{N}} = g^0_{\bar{N}})$ obtained by the supergravity q-map consists of complete QK metrics.

Classification of complete PSR curves and surfaces

Thorem [CHM]

There are only 2 complete PSR curves (up to equivalence):

i) $\{(x, y) \in \mathbb{R}^2 | x^2 y = 1,\ x > 0\}$,

ii) $\{(x, y) \in \mathbb{R}^2 | x(x^2 - y^2) = 1,\ x > 0\}$.

Thorem [CDL]

There are only 5 discrete examples and a 1-parameter family of complete PSR surfaces:

a) $\{(x, y, z) \in \mathbb{R}^3 | xyz = 1,\ x > 0, y > 0\}$,

b) $\{(x, y, z) \in \mathbb{R}^3 | x(xy - z^2) = 1,\ x > 0\}$,

c) $\{(x, y, z) \in \mathbb{R}^3 | x(yz + x^2) = 1,\ x < 0, y > 0\}$,

d) $\{(x, y, z) \in \mathbb{R}^3 | z(x^2 + y^2 - z^2),\ z < 0\}$,

e) $\{(x, y, z) \in \mathbb{R}^3 | x(y^2 - z^2) + y^3 = 1,\ y < 0, x > 0\}$,

f) $\{\cdots | y^2 z - 4x^3 + 3xz^2 + bz^3 = 1,\ z < 0, 2x > z\}$, $b \in (-1, 1)$.

Classification of complete PSR manifolds with reducible cubic polynomial

Theorem [Jüngling, Lindemann, MSC theses Hamburg]

Every complete PSR manifold $\mathcal{H} \subset \{h = 1\} \subset \mathbb{R}^{n+1}$ for which h is reducible is linearly equivalent to one of the following:

a) $\{x_{n+1}(\sum_{i=1}^{n-1} x_i^2 - x_n^2) = 1,\ x_{n+1} < 0, x_n > 0\}$,

b) $\{(x_1 + x_{n+1})(\sum_{i=1}^{n} x_i^2 - x_{n+1}^2) = 1, \quad x_1 + x_{n+1} < 0\}$,

c) $\{x_1(\sum_{i=1}^{n} x_i^2 - x_{n+1}^2) = 1, \quad x_1 < 0, x_{n+1} > 0\}$,

d) $\{x_1(x_1^2 - \sum_{i=2}^{n+1} x_i^2) = 1, \quad x_1 > 0\}$.

Completeness of centroaffine hypersurfaces

Let $\mathcal{H} \subset \mathbb{R}^{n+1}$ be a centroaffine hypersurface with positive definite centroaffine metric g.

We are interested in the relation between

1) closedness,
2) Euclidian completeness and
3) completeness (with respect to g).

Under natural assumptions:
3) $\Longrightarrow$ 1) $\iff$ 2).

Main problem:

Prove that 1) $\Longrightarrow$ 3) in some interesting cases.

Example: Theorem (Cheng and Yau, CPAM '89)

1) $\Longrightarrow$ 3) if $\mathcal{H}$ is an affine sphere, i.e. if $\nabla^g \nu = 0$.

Completeness of higher dimensional PSR manifolds

Theorem [CNS]

A PSR manifold $\mathcal{H} \subset \{h = 1\} \subset \mathbb{R}^{n+1}$ is complete if and only if $\mathcal{H} \subset \mathbb{R}^{n+1}$ is closed.

Corollary

Let $\mathcal{H}$ be a locally strictly convex component of the level set $\{h = 1\}$ of a homogeneous cubic polynomial h on $\mathbb{R}^{n+1}$. Then $\mathcal{H}$ defines a complete quaternionic Kähler metric of negative scalar curvature on $\mathbb{R}^{4n+8}$.

Open problem

Does the theorem extend to (definite) centroaffine hypersurfaces defined by homogeneous polynomials h of higher degree?

State of the art:

1. It holds for generic polynomials.
2. It does not hold for general (real analytic) functions.

Sketch of proof of the main theorem I

- Let $\mathcal{H} \subset \mathbb{R}^{n+1}$ be a Euclidian complete centroaffine hypersurface with positive definite centroaffine metric g.
- We have to show that $\mathcal{H}$ is complete if $\mathcal{H} \subset \{h = 1\}$ for a homogeneous cubic polynomial h. Let us not assume this yet.
- Consider the open cone $U = \mathbb{R}^{>0} \cdot \mathcal{H} \subset \mathbb{R}^{n+1}$ and let $k \in \mathbb{R}^*$.

Lemma 1

- There exists a unique smooth homogeneous function $h : U \to \mathbb{R}$ of degree k such that $h|_{\mathcal{H}} = 1$.
- For every hyperplane E tangent to $\mathcal{H}$ the intersection $B := U \cap E \subset E$ is a bounded convex domain.
- $$\varphi : B \to \mathcal{H}, \quad x \mapsto h(x)^{-1/k} x,$$

  is a parametrization of $\mathcal{H}$.

Sketch of proof of the main theorem II

Lemma 2

In the above parametrization the centroaffine metric is given

$$g = -\frac{1}{k\bar{h}}\partial^2\bar{h} + \frac{k-1}{(k\bar{h})^2}d\bar{h}^2,$$

where $\bar{h}$ denotes the restriction of h to B and ∂ denotes the flat connection of the affine space $E \supset B$.

Lemma 3

Let $k > 0$. Assume that there exists $\varepsilon \in (0, k)$ such that $f = \sqrt[k-\varepsilon]{\bar{h}}$ is concave. Then $\mathcal{H}$ is complete.

Sketch of pf. of Lemma 3

A calculation shows

$$g = \frac{k-\varepsilon}{f}\left(-\frac{1}{k}\partial^2 f\right) + \frac{\varepsilon}{(k-\varepsilon)(k\bar{h})^2}d\bar{h}^2 \geq \underbrace{\frac{\varepsilon}{k^2(k-\varepsilon)}}_{C:=}(d\ln\bar{h})^2.$$

Sketch of proof of the main theorem III

Let $\gamma : I = [0, T) \to B,\ T \in (0, \infty]$, be a curve which is not contained in any compact subset of B and $I \ni t_i \to T$.

- Then $h(\gamma(t_i)) \to 0$ and the previous estimate implies

$$\begin{aligned} L(\gamma) &\geq L(\gamma|_{[0,t_i]}) \geq C \int_0^{t_i} \left| \frac{d}{dt} \ln h \circ \gamma \right| dt \geq C \left| \int_0^{t_i} \frac{d}{dt} \ln h \circ \gamma \right| dt \\ &= C |\ln h(\gamma(t_i)) - \ln h(\gamma(0))| \to \infty \quad \square \end{aligned}$$

Lemma 4
If h is a cubic polynomial then $\sqrt{h}$ is concave

Lemma 4 shows that the assumptions of Lemma 3 are satisfied with $(k, \epsilon) = (3, 1)$. This finishes the proof of the main theorem. $\square$

Further results (about general centroaffine hypersurfaces): The canonical Lorentzian metric on the open cone U

Proposition

Let $\mathcal{H} \subset \mathbb{R}^{n+1}$ be any Euclidian complete centroaffine hypersurface with positive definite centroaffine metric , $k > 1$ and h the corresponding homogeneous function of degree k. Then

$$g_L := -\frac{1}{k}\partial^2 h$$

is a Lorentzian metric on U, which is globally hyperbolic iff $\mathcal{H}$ is complete.

Further results: Regular boundary behaviour implies completeness

Let $\mathcal{H} \subset \mathbb{R}^{n+1}$ be any Euclidian complete centroaffine hypersurface with positive definite centroaffine metric. We assume that $k > 1$ and that h extends to a smooth homogeneous function $h : V \to \mathbb{R}$ defined on some open subset $V \subset \mathbb{R}^{n+1}$ such that $\overline{U} \setminus \{0\} \subset V$.

Definition

Under the above assumptions, we say that the hypersurface $\mathcal{H}$ has regular boundary behaviour if

(i) $dh_p \neq 0$ for all $p \in \partial U \setminus \{0\}$. In particular, $\partial U \setminus \{0\}$ is smooth.

(ii) $-\partial^2 h$ is positive semi-definite on $T(\partial U \setminus \{0\})$ with only one-dimensional kernel.

Regular boundary behaviour implies completeness

Theorem [CNS]

Let $\mathcal{H} \subset \mathbb{R}^{n+1}$ be a Euclidian complete centroaffine hypersurface with regular boundary behaviour. Then $\mathcal{H}$ is complete.

Regular boundary behaviour is generic

- Let $V \subset \mathbb{R}^{n+1}$ be an open subset and $k > 1$.
- Denote by $\mathcal{F}(V,k) \subset C^\infty(V)$ the set of homog. fcts. h of deg. k s.t. $\exists$ open cone $U \subset V$ s.t. $\overline{U} \setminus \{0\} \subset V$ and s.t.

$$\mathcal{H}(h,U) := \{p \in U | h(p) = 1\}$$

is Euclidian complete with $g > 0$.
- Put
$\mathcal{F}_{reg}(V,k) := \{h \in \mathcal{F} | \mathcal{H}(h,U)$ has reg. bdry. beh. for some $U\}$.

Theorem (CNS)

$\mathcal{F}_{reg}(V,k) \subset \mathcal{F}(V,k)$ is open and dense (in the Fréchet topology).

Regular boundary behaviour is generic: Case of polynomial functions

- Denote by

$$\mathcal{P}(k) \subset \mathcal{F}(\mathbb{R}^{n+1}, k), \quad \mathcal{P}_{reg}(k) = \mathcal{F}_{reg}(\mathbb{R}^{n+1}, k)$$

the subsets consisting of polynomial functions.

Theorem [CNS]

$\mathcal{P}_{reg}(k) \subset \mathcal{P}(k)$ is open and dense.