# Beirut Lectures II: Geometric Constructions Relating Different Special Geometries

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Conference on Differential Geometry, Beirut, 2015

# Recap of Lecture I

Scalar geometry of $N = 2$ theories depends on: space-time dimension d and field content: vector multiplets or hypermultiplets

## Special geometries of rigid $N = 2$ supersymm. theories

| d | vector multiplets | hypermultiplets |
|---|---|---|
| 5 | affine special real | hyper-Kähler |
| 4 | affine special Kähler | hyper-Kähler |
| 3 | hyper-Kähler | hyper-Kähler |

## Special geometries of $N = 2$ supergravity theories

| d | vector multiplets | hypermultiplets |
|---|---|---|
| 5 | projective special real | quat. Kähler |
| 4 | projective special Kähler | quat. Kähler |
| 3 | quaternionic Kähler | quat. Kähler |

# Plan of the second lecture

Constructions induced by dimensional reduction:

- rigid r-map
- rigid c-map
- supergravity r-map
- supergravity c-map
- global properties of these constructions

More general constructions:

- one loop quantum corrections of supergravity c-map metrics
- HK/QK-correspondence

# Some references for Lecture II


[ACDM] Alekseevsky, C.– , Dyckmanns, Mohaupt (JGP ‘15).

[H13] Hitchin (CMP ‘13).

[ACM] Alekseevsky, C.– , Mohaupt (CMP ‘13).

[CHM] C.–, Han, Mohaupt (CMP ‘12).

[APP] Alexandrov, Persson, Pioline (JHEP ‘11).

[H] Hitchin (PIM ‘09).

[AC] Alekseevsky, C.– (CMP ‘09).

[Ha] Haydys (JGP ‘08).

[RSV] Robles-Llana, Saueressig, Vandoren (JHEP ‘06).

[CMMS] C.–, Mayer, Mohaupt, Saueressig (JHEP ‘04).

[ACD] Alekseevsky, C.– , Devchand (JGP ‘02).

[BC] Baues, C.– (PLMS ‘01).

[AC00] Alekseevsky, C.– (PLMS ‘00).

[L] Z. Lu (MA ‘99).

[C] C.– (TAMS ‘98).

[DV92] de Wit, Van Proeyen (CMP ‘92).

[CFG] Ferrara, Sabharwal (NPB ‘90).

[CFG] Cecotti, Ferrara, Girardello (IJMP ‘89).

# The rigid r-map I: from affine special real to affine special Kähler manifolds

## Dimensional reduction from 5 to 4 space-time dimensions

- It was observed by de Wit and Van Proeyen that dim. reduction of sugra coupled to vector multiplets from 5 to 4 space-time dimensions relates the scalar geometries by a construction called the supergravity r-map [DV92].
- The analogous construction for theories without gravity is called the rigid r-map [CMMS].
- It relates affine special real manifolds to affine special Kähler manifolds
- and has the following geometric description [AC].

# The rigid r-map II

## Geometric description of the rigid r-map

- Let $(M, \nabla, g)$ be an intrinsic ASR mf. Consider the tangent bdl. $\pi : N = TM \to M$.
- Using the flat connection $\nabla$ we can canonically identify

$$TN = T^h N \oplus T^v N \cong \pi^* TM \oplus \pi^* TM.$$

- Therefore

$$J := \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix},$$

  defines an almost cx. structure on $N$.
- Similarly,

$$g_N := \begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix}$$

  defines a Riem. metric on $N$.

# The rigid r-map III

## Geometric description of the rigid r-map continued

- Next we define a (1, 2)-tensor field $S^N$ on $N$ by

$$S^N_{X^h} := \begin{pmatrix} S_X & 0 \\ 0 & -S_X \end{pmatrix}, \quad S^N_{X^v} := \begin{pmatrix} 0 & -S_X \\ -S_X & 0 \end{pmatrix},$$

  where $S = D - \nabla$, $D = L.C.$, and $X^h$, $X^v$ are the hor. and vert. lifts of $X \in \mathfrak{X}(M)$.
- Finally we define a connection $\nabla^N := D^N - S^N$ on $N$, where $D^N = L.C.$

### Theorem

Let $(M, \nabla, g)$ be an affine special real manifold. Then $(N, g_N, J, \nabla^N)$ is an affine special Kähler manifold.

- The correspondence $(M, \nabla, g) \mapsto (N, g^N, J, \nabla^N)$ is called the rigid r-map.

# The rigid c-map I: from affine special Kähler to hyper-Kähler manifolds

## Dimensional reduction of vector multiplets from 4 to 3 space-time dimensions

- As observed by Cecotti, Ferrara and Girardello, dim. reduction of $N = 2$ vector multiplets from 4 to 3 space-time dimensions relates the corresponding scalar geometries by a construction called the rigid c-map [CFG].
- It relates affine special Kähler to hyper-Kähler manifolds
- and has the following geometric description [ACD].

# The rigid c-map II

## Geometric description of the rigid c-map

- Let $(M, J, g, \nabla)$ be an affine special Kähler manifold. Consider the cotangent bdl. $\pi : N = T^*M \to M$.
- Using the flat connection $\nabla$ we can canonically identify

$$TN = T^h N \oplus T^v N \cong \pi^* TM \oplus \pi^* T^* M.$$

- Therefore

$$g_N = \begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix}$$

  defines a Riem. metric on $N$ and
-

$$J_1 = \begin{pmatrix} J & 0 \\ 0 & J^* \end{pmatrix}, \; J_2 = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}$$

  define two almost cx. structures on $N$.

# The rigid c-map III

## Geometric description of the rigid c-map continued

**Theorem**

Let $(M, J, g, \nabla)$ be an affine special Kähler manifold. Then $(N, g_N, J_1, J_2, J_3 = J_1 J_2)$ is a hyper-Kähler manifold.

- The correspondence $(M, J, g, \nabla) \mapsto (N, g_N, J_1, J_2, J_3 = J_1 J_2)$ is called the rigid c-map.

# The supergravity r-map I: from projective special real to projective special Kähler manifolds

- The sugra r-map of [DV] can be described as follows [CHM]:
- Let $\mathcal{H} \subset \mathbb{R}^{n+1}$ be a PSR mf. and $h$ the corresponding cubic polynomial.
- Then $U = \mathbb{R}^{>0} \cdot \mathcal{H} \subset \mathbb{R}^{n+1}$ is an open cone.
- We endow it with the Riem. metric

$$g_U = -\frac{1}{3}\partial^2 \ln h,$$

isometric to the product metric $dt^2 + g_{\mathcal{H}}$ on $\mathbb{R} \times \mathcal{H}$

- and finally the domain $\bar{M} = U \times \mathbb{R}^{n+1}$ with the Riem. metr.

$$g_{\bar{M}} := \frac{3}{4} \sum_{a,b=1}^{n+1} g_{ab}(dx^a dx^b + dy^a dy^b), \quad g_{ab} := g_U\left(\frac{\partial}{\partial x^a}, \frac{\partial}{\partial x^b}\right).$$

# The supergravity r-map II

Theorem

(i) $(\bar{M}, g_{\bar{M}})$ *defined above is projective special Kähler with respect to the cx. structure $J$ defined by the embedding*

$$\bar{M} = U \times \mathbb{R}^{n+1} \to \mathbb{C}^{n+1}, \quad (x, y) \mapsto y + ix.$$

(ii) *The natural inclusions $\mathcal{H} \subset U \cong U \times \{0\} \subset \bar{M}$ are isometric and totally geodesic.*

- The correspondence $\mathcal{H} \mapsto (\bar{M}, J, g_{\bar{M}})$ is called the supergravity r-map.
- It maps PSR mfs. of dim. n to PSK mfs. of (real) dim. 2n+2.

# The supergravity c-map I: from projective special Kähler to quaternionic Kähler manifolds

- Dim. reduction of sugra coupled to vector multiplets from 4 to 3 space-time dimensions relates the corresponding scalar geometries by a construction called the supergravity c-map.
- The resulting quaternionic Kähler metric $g_{FS}$ was computed by Ferrara and Sabharwal [FS], cf. [H,CHM, ...].
- Here we follow [CHM]: In the case of a PSK domain $(\bar{M}, g_{\bar{M}})$ of dim. $2n$ the metric $g_{FS}$ has the following structure:

$$g_{FS} = g_{\bar{M}} + g_G,$$

  where $g_G$ is a family of left-invariant Riemannian metrics on $G = \mathrm{Iwa}(\mathrm{SU}(n+2,1))$ depending on $p \in \bar{M}$.
- In particular, $g_{FS}$ is defined on the product $\bar{N} := \bar{M} \times G$.
- The inclusion $\bar{M} \cong \bar{M} \times \{e\} \subset \bar{N}$ is totally geodesic.

# The supergravity c-map II

## The explicit form of the family of metrics $(g_G(p))_{p\in M}$:

$$\frac{1}{4\phi^2}d\phi^2 + \frac{1}{4\phi^2}\left(d\tilde{\phi} + \sum(\zeta^i d\tilde{\zeta}_i - \tilde{\zeta}_i d\zeta^i)\right)^2 + \frac{1}{2\phi}\sum \mathcal{I}_{ij}(p)d\zeta^i d\zeta^j$$

$$+\frac{1}{2\phi}\sum \mathcal{I}^{ij}(p)\left(d\tilde{\zeta}_i + \sum \mathcal{R}_{ik}(p)d\zeta^k\right)\left(d\tilde{\zeta}_j + \sum \mathcal{R}_{j\ell}(p)d\zeta^\ell\right),$$

- where $(\phi, \tilde{\phi}, \zeta^1, \ldots, \zeta^{n+1}, \tilde{\zeta}_1, \ldots, \tilde{\zeta}_{n+1}) : G \to \mathbb{R}^{>0} \times \mathbb{R}^{2n+3}$ is a global coord. system on $G \cong \mathbb{R}^{2n+4}$ and
- $\mathcal{R}_{ij}$, $\mathcal{I}_{ij}$ are real and imaginary parts of

$$\bar{F}_{ij} + \sqrt{-1}\frac{\sum N_{ik}z^k \sum N_{j\ell}z^\ell}{\sum N_{kl}z^k z^\ell},$$

- determined by the prepot. $F$ of the underlying CASK dom.
- $\mathcal{I} = (\mathcal{I}_{ij}) > 0$ [CHM]. Hence $(\mathcal{I}^{ij}) = \mathcal{I}^{-1}$ is defined and $g_G > 0$.

# The supergravity c-map III

## Geometric interpretation of the fiber metric

- $(G, g_G(p))$ is isometric to $\mathbb{C}H^{n+2}$.
- The principal part of

$$g_G = \frac{1}{4\phi^2} d\phi^2 + \frac{1}{4\phi^2} \left( d\tilde{\phi} + \sum (\zeta^i d\tilde{\zeta}_i - \tilde{\zeta}_i d\zeta^i) \right)^2 + \frac{1}{2\phi} g_G^{pr}$$

  is related to the CASK domain $\pi : M \to \bar{M}$ as follows:
- $M$ has a can. realization [ACD] as a Lagrangian cone in $V = (\mathbb{C}^{2n+2}, \Omega, \gamma)$, where $g_M = \mathrm{Re}\, \gamma|_M$ is induced.
- Therefore we have a hol. map $\bar{M} \to Gr_0^{1,n}(V) = Sp(\mathbb{R}^{2n+2})/U(1,n)$, $p \mapsto L_p$.
- Composing it with the $Sp(\mathbb{R}^{2n+2})$-equivariant embedding

$$Gr_0^{1,n}(V) \to Sym^1_{2,2n}(\mathbb{R}^{2n+2}) = SL(2n+2, \mathbb{R})/SO(2, 2n)$$

  we obtain $p \mapsto (g_{IJ}(p)) \in Sym^1_{2,2n}(\mathbb{R}^{2n+2})$.

# The supergravity c-map IV

## Geometric interpretation of the fiber metric continued

- In fact, $\sum g_{IJ}(p) dq^I dq^J = g_M(\tilde{p})$, $\forall \tilde{p} \in \pi^{-1}(p)$, where $(q^I)_{I=1,\ldots,2n+2} = (x^i = \mathrm{Re}\, z^i, y_j = \mathrm{Re}\, F_j)_{i,j=0,\ldots,n}$.
- Next we change the indefinite scalar product $(g_{IJ}(p))$ to $(\hat{g}_{IJ}(p)) > 0$ by means of an $Sp(\mathbb{R}^{2n+2})$-equivariant diffeo. $\psi : F_0^{1,n}(V) \to F_0^{n+1,0}(V)$ from Griffiths to Weil flags.
- In the case of the $CY_3$ moduli space this is related to the switch from Griffiths to Weil intermediate Jacobians [C,H]
- This corresponds to switching the sign of the indefinite metric $g_M$ on the negative definite distribution $\mathcal{D}^\perp$.
- We show that the cx. symm. matrix $\mathcal{R} + i\mathcal{J} \in Sym_{n+1,0}(\mathbb{C}^{n+1})$ corresponds to the pos. def. Lagrangian subspace $L'$ defined by $\psi(\ell, L) = (\ell, L')$, where $L = L_p$ and $\ell = p = \mathbb{C}\tilde{p}$. This proves $\mathcal{J} > 0$.
- Finally we prove that $g_G^{pr}(p) = \sum \hat{g}^{IJ}(p) dq_I dq_J$, where $(q_I) = (\tilde{\zeta}_i, \zeta^j)$.

# The supergravity c-map V

## Concluding remark

- In the general case, when the PSK mf. $\bar{M}$ is covered by PSK domains, we show that the local Ferrara-Sabharwal metrics are consistent and define a QK mf. $\bar{N}$ which fibers over $\bar{M}$ as a bundle of groups with totally geodesic can. section $\bar{M} \hookrightarrow \bar{N}$.
- This shows that the supergravity $c$-map is globally defined for every PSK mf.

# Global properties of the r- and c-maps

## Theorem [CHM]

(i) The supergravity r-map maps complete PSR mfs. $\mathcal{H}$ of dim. $n$ to complete PSK mfs. $\bar{M}$ of dim. 2n+2.

(ii) The supergravity c-map maps complete PSK mfs. $\bar{M}$ of dim. $2n$ to complete QK mfs. $\bar{N}$ of dim. $4n+4$ and $Ric < 0$.

(iii) There are totally geodesic inclusions $\mathcal{H} \subset \bar{M}$ and $\bar{M} \subset \bar{N}$ in (i) and (ii), respectively.

## Remarks

- The same results hold for the rigid r- and c-map but
- $\nexists$ nonflat complete ASK mfs. [L], as follows [BC] from the Calabi-Pogorelov thm., and, hence, no nonflat ASR mfs.
- $\exists$ examples of homogeneous and, hence, complete PSR and PSK mfs. See [DV92,AC00] for some classification results.

# One-loop correction of the FS-metric I

Consider the FS-metric associated with a PSK domain $\bar{M}$. The following symmetric tensor field is called one loop correction of the FS-metric [RSV]:

$$
\begin{aligned}
g_{FS}^{c} &= \frac{\phi + c}{\phi} g_{\bar{M}} + \frac{1}{4\phi^2} \frac{\phi + 2c}{\phi + c} d\phi^2 \\
&\quad + \frac{1}{4\phi^2} \frac{\phi + c}{\phi + 2c} \left( d\tilde{\phi} + \sum (\zeta^j d\tilde{\zeta}_j - \tilde{\zeta}_j d\zeta^j) + ic(\bar{\partial} - \partial)\mathcal{K} \right)^2 \\
&\quad + \frac{1}{2\phi} \sum dq_a \hat{g}^{ab} dq_b + \frac{2c}{\phi^2} e^{\mathcal{K}} \left| \sum (X^j d\tilde{\zeta}_j + F_j(X) d\zeta^j) \right|^2,
\end{aligned}
$$

where $c \in \mathbb{R}$, $X^j = z^j / z^0$ and

$$
\mathcal{K} = -\log \left( \sum X^i N_{ij} \bar{X}^j \right)
$$

is the Kähler potential for the projective special Kähler metric $g_{\bar{M}}$.

# One-loop correction of the FS-metric II

## Theorem [ACDM]

For $c \geq 0$, the one loop correction $g_{FS}^c$ defines a 1-parameter family of quaternionic Kähler metrics on $\bar{N} = \bar{M} \times G$ deforming the FS-metric $g_{FS} = g_{FS}^0$.

## Sketch of proof

- Applying the rigid c-map to the underlying CASK mf. $M$ we obtain a pseudo-HK mf. $N$.
- The $\nabla$-horizontal lift of $2J\xi$ defines a Killing v.f. $Z$ on $N$ satisfying the assumptions of the HK/QK-correspondence explained on the next slides.
- Applying the HK/QK-correspondence yields a 1-parameter family of pseudo-QK metrics, of which we determine the domain of positivity.
- Finally we check that this family coincides with the one loop correction of the FS-metric. $\square$

# The HK/QK-correspondence I

The following result generalizes work of Haydys [Ha]:

### Theorem [ACM]

- Let $(M, g, J_1, J_2, J_3)$ be a pseudo-HK mf. with a timelike or spacelike Killing v.f. $Z$ s.t.
  - $\mathcal{L}_Z J_1 = 0$, $\mathcal{L}_Z J_2 = -2J_3$,
  - $\exists f : df = -\omega_1(Z, \cdot)$, $\omega_1 = g(J_1 \cdot, \cdot)$,
  - $f$ and $f_1 := f - g(Z, Z)/2$ are nowhere zero.

  Then from the data $(M, g, J_1, J_2, J_3, f)$ one can construct a pseudo-QK mf. $(M', g')$ with $\dim M' = \dim M$. The signature of $g'$ depends only on that of $g$ and the signs of $f$ and $f_1$.
- Cases when $g' > 0$:
- $g' > 0$ of $Ric > 0$ if $g > 0$ and $f_1 > 0$ and
- $g' > 0$ of $Ric < 0$ if either:
  $g > 0$ and $f < 0$ or
  $g$ has signature $(4k, 4)$, $f < 0$ and $f_1 > 0$.

# The HK/QK-correspondence II

## Remarks

- In [ACDM] we give a simple explicit formula for the QK-metric $g'$ obtained from the HK/QK-correspondence.
- Using this formula, we check that the rigid c-map metric is mapped to the one loop corrected supergravity c-map metric by this correspondence.
- A similar result was obtained in [APP] by applying twistor methods and the inverse construction, the QK/HK-correspondence.
- The simplest case of the construction is $\bar{M} = \{pt\}$. In this case, we obtain a 1-parameter deformation of $\mathbb{C}H^2$ by explicit complete QK metrics. (The full domain of positivity of the one-loop correction has also components with incomplete metric, including a metric found by Haydys [Ha].)
- This example of the HK/QK-correspondence is also discussed in [Hi13], but without the QK metric.