Beirut Lectures I: Special Geometry

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Plan of the mini course

I. Special geometry
II. Geometric constructions relating different special geometries
III. Constructions of complete quaternionic Kähler manifolds

Plan of the first lecture

- Physical motivation of special geometry
- Affine and projective special real geometry
- Affine and projective special Kähler geometry
- Hyper-Kähler and quaternionic Kähler geometry

Some references for Lecture I

[CNS] C.–, Nardmann, Suhr (CAG, accepted), math.DG:1407.3251.

[AC] Alekseevsky, C.– (CMP '09).

[CM] C.–, Mohaupt (JHEP '09).

[CMMS] C.–, Mayer, Mohaupt, Saueressig (JHEP '04).

[ACD] Alekseevsky, C.– , Devchand (JGP '02).

[F] Freed (CMP '99).

[L] LeBrun (Duke '91).

[GST] Günaydin, Sierra, Townsend (NPB '84).

[DV] de Wit, Van Proeyen (NPB '84).

[BW] Bagger, Witten (NPB '83)

Physical motivation

Scalar geometry

$$\mathcal{L} = -\sum g_{ij}(\phi^1, \ldots, \phi^n) h^{\mu\nu} \partial_\mu \phi^i \partial_\nu \phi^j + \ldots$$

Physics Definition

Special geometry is the scalar geometry of supersymmetric field theories with 8 real supercharges.

One distinguishes between

- Affine special geometry/supersymmetric gauge theories and
- Projective special geometry/supergravity theories.

Affine special real manifolds I: extrinsic and intrinsic definition

Definition

An affine special real (ASR) manifold is a domain $M \subset \mathbb{R}^n$ endowed with a Riemannian metric $g = \partial^2 h$, which is the Hessian of a cubic polynomial.

Definition

An intrinsic ASR manifold (M, ∇, g) is a Riemannian manifold (M, g) endowed with a flat torsion-free connection ∇ such that ∇g is completely symmetric and ∇-parallel.

Remark

ASR mfs. = scalar mfs. of 5d vector multiplets [CMMS]

Affine special real manifolds II: intrinsic characterization

Theorem [AC]

(i) Let (M, g) be an n-dim. ASR manifold with flat connection ∇ induced from the inclusion $M \subset \mathbb{R}^n$. Then (M, ∇, g) is an intrinsic ASR manifold.

(ii) Conversely, let (M, g, ∇) be an n-dim. s.c. intrinsic ASR manifold. Then there exists an affine immersion $\varphi : (M, \nabla) \to \mathbb{R}^n$ and a cubic polynomial h s.t. $g = \varphi^* \partial^2 h$. In particular, $(\varphi(M), \partial^2 h)$ is an ASR manifold. (φ is unique up to affine transformations of $\mathbb{R}^n$.)

Projective special real manifolds I: extrinsic definition

Definition

A projective special real (PSR) manifold is a hypersurface $\mathcal{H} \subset \mathbb{R}^{n+1}$ s.t. $\exists$ homog. cubic polynomial h on $\mathbb{R}^{n+1}$ s.t.

i) $h = 1$ on $\mathcal{H}$ and

ii) $\partial^2 h$ is negative definite on $T\mathcal{H}$.

$\mathcal{H}$ is endowed with the Riemannian metric

$$g_{\mathcal{H}} = -\frac{1}{3}\iota^* \partial^2 h,$$

where $\iota : \mathcal{H} \to \mathbb{R}^{n+1}$ is the inclusion map.

$\mathcal{H}$ complete $:\Longleftrightarrow$ $(\mathcal{H}, g_{\mathcal{H}})$ complete.

Remark

PSR mfs. = scalar mfs. of 5d sugra coupled to vector multpl. [GST]

Centroaffine structures

Definition

A centroaffine structure on a smooth manifold M is a triple (∇, g, ν) consisting of a torsion-free connection ∇, a pseudo-Riemannian metric g and a volume form ν such that

(i) $\nabla \nu = 0$,

(ii) the curvature R of ∇ is given by

$$R(X, Y)Z = -(g(Y, Z)X - g(X, Z)Y)$$

for all $X, Y, Z \in \mathfrak{X}(M)$ and

(iii) ∇g is completely symmetric.

(M, ∇, g, ν) is called a centroaffine manifold.

Induced centroaffine structure on a hypersurface

Let $\mathcal{H} \subset (\mathbb{R}^{n+1}, \det)$ be a centroaffine hypersurface, i.e. a nondeg. hypersurface transversal to the position vector field ξ.

Then

(i) $\nu = \iota_\xi \det$ is a volume form on $\mathcal{H}$ and

(ii) the (affine) Gauß equation

$$\partial_X Y = \nabla_X Y + g(X, Y)\xi, \quad X, Y \in \mathfrak{X}(\mathcal{H}),$$

defines a centroaffine structure (∇, g, ν).

Remarks

1. The metric g is called the centroaffine metric.
2. For a PSR manifold $(\mathcal{H}, g_{\mathcal{H}})$ we have $g = g_{\mathcal{H}}$.

Projective special real manifolds II: intrinsic definition

Definition

An intrinsic PSR manifold is a centroaffine manifold (M, ∇, g, ν) with $g > 0$ such that the covariant derivative of the cubic form $C = \nabla g$ is given by

$$(\nabla_X C)(Y, Z, W) =$$
$$g(X, Y)g(Z, W) + g(X, Z)g(W, Y) + g(X, W)g(Y, Z),$$

for all $X, Y, Z, W \in \mathfrak{X}(M)$.

Remark

The above equation implies that ∇C is totally symmetric, that is a quartic form.

Projective special real manifolds III: intrinsic characterization

Theorem [CNS]

(i) Let $\mathcal{H} \subset \mathbb{R}^{n+1}$ be a PSR manifold with its induced centroaffine structure (∇, g, ν). Then $(\mathcal{H}, \nabla, g, \nu)$ is an intrinsic PSR manifold.

(ii) Conversely, let (M, ∇, g, ν) be a s.c. intrinsic PSR manifold. Then there exists an embedding $\varphi : M \to \mathbb{R}^{n+1}$ such that $\mathcal{H} := \varphi(M) \subset \mathbb{R}^{n+1}$ is a PSR manifold.

The embedding φ is unique up to linear unimodular transformations of $\mathbb{R}^{n+1}$.

Affine special Kähler manifolds

Definition

A (pseudo-) Kähler manifold (M, g, J) is a (pseudo-) Riemannian manifold (M, g) endowed with a parallel skew-symm. cx. str. J.

Definition [F]

An affine special (pseudo-) Kähler manifold (M, J, g, ∇) is a (pseudo-) Kähler mf. (M, J, g) endowed with a flat torsionfree connection ∇ such that

(i) $\nabla\omega = 0$, where $\omega = g(\cdot, J\cdot)$,

(ii) $d^\nabla J = 0$, where J is considered as a 1-form with values in TM.

Remark

Affine special Kähler mfs. = scalar mfs. of 4d N=2 vector multiplets,
Projective special Kähler mfs. = scalar mfs. of 4d supergravity coupled to N=2 vector multiplets [DV].

Conical and projective special Kähler manifolds

Definition [ACD, CM]

A conical affine special Kähler (CASK) manifold (M, J, g, ∇, ξ) is an affine special (pseudo-)Kähler manifold (M, J, g) endowed with a vector field ξ such that

(iii) $\nabla\xi = D\xi = \mathrm{Id}$, where D is the Levi Civita connection and

(iv) g is positive definite on $\mathcal{D} := \mathrm{span}\{\xi, J\xi\}$ and negative definite on $\mathcal{D}^{\perp}$.

$\Rightarrow$ ξ and $J\xi$ generate a hol. action of a 2-dim. Abelian Lie algebra. We will assume that the action lifts to a principal $\mathbb{C}^*$-action with the base $\bar{M} = M/\mathbb{C}^*$. Then $J\xi$ generates a free isometric and Hamiltonian S^1-action and $\bar{M}$ inherits a Kähler metric $\bar{g}$. $(\bar{M}, \bar{g})$ is called a projective special Kähler (PSK) manifold.

Extrinsic construction of special Kähler manifolds I

The ambient space

$V = (\mathbb{C}^{2n}, \Omega, \tau)$, $\Omega = \sum dz^i \wedge dw_i$, $\tau =$ cx. conjugation.
$\rightarrow$ pseudo-Hermitian form $\gamma := \sqrt{-1}\Omega(\cdot, \tau\cdot)$.

Definition

A holomorphic immersion $\phi : M \to V$ is called nondegenerate if $\phi^*\gamma$ is nondeg. It is called Lagrangian if $\phi^*\Omega = 0$ and $\dim M = n$.

Theorem [ACD]

- A nondeg. hol. Lagrangian immersion $\phi : M \to V$ induces an affine special pseudo-Kähler structure (J, g, ∇) on M.
- Every s.c. affine special (pseudo-) Kähler mf. (M, J, g, ∇) of dim. n admits a nondeg. Lagr. immersion $\phi : M \to V$ inducing (J, g, ∇) on M. The immersion is unique up to affine transformations with real symplectic linear part.

Extrinsic construction of special Kähler manifolds II

Example (affine special pseudo-Kähler domains)

Let F be a holomorphic function defined on a domain $M \subset \mathbb{C}^n$ such that the matrix

$$(N_{ij}) = (2\mathrm{Im}\, F_{ij}),$$

is nondeg, where $F_i = \frac{\partial F}{\partial z^i}$, $F_{ij} = \frac{\partial F}{\partial z^i \partial z^j}$ etc. Then

$$\phi : M \to V, \quad z = (z^1, \ldots, z^n) \mapsto (z, F_1, \ldots, F_n)$$

is a nondeg. Lagr. immersion and, thus, induces an affine special pseudo-Kähler (J, g, ∇) structure on M.

Definition

Affine special pseudo-Kähler manifolds as in the above example are called affine special pseudo-Kähler domains. The function F is called a holomorphic prepotential.

Extrinsic construction of special Kähler manifolds III

Since every Lagrangian submanifold of (V, Ω) is locally defined by equations $w_i = F_i(z)$, $i = 1, \ldots, n$, for some hol. function F and some choice of adapted coordinates (z^i, w_i), we obtain:

Corollary

Let (M, J, g, ∇) be an affine special pseudo-Kähler manifold. Then for every $p \in M$ there exists a neighborhood U isomorphic to an affine special pseudo-Kähler domain.

Remark

Similar results hold for conical and projective special Kähler manifolds. CASK manifolds are realized as conical hol. nondeg. Lagrangian immersions. The corresponding prepotential is defined on a $\mathbb{C}^*$-invariant domain $M \subset \mathbb{C}^n$ and is required to be homogeneous of degree 2 and to satisfy: $\sum N_{ij} z^i \bar{z}^j > 0$ and the real symmetric matrix (N_{ij}) has signature $(1, n)$ on M.

Extrinsic construction of special Kähler manifolds IV

Example (complex hyperbolic space as PSK domain)

$$F = \frac{i}{4}\left((z^0)^2 - \sum_{j=1}^{n}(z^j)^2\right)$$

on $M = \{|z^0|^2 - \sum_{j=1}^{n}|z^j|^2 > 0\} \subset \mathbb{C}^{n+1}$ is a prepot. for a CASK domain (M, J, g, ∇, ξ). The corresponding PSK domain is $\mathbb{C}H^n$.

Hyper-Kähler manifolds

Definition

A (pseudo-) hyper-Kähler manifold (M, g, J_1, J_2, J_3) is a (pseudo-) Riemannian manifold (M, g) endowed with 3 pairwise anticomm. parallel skew-symm. cx. structures J_1, J_2, J_3 s.t. $J_3 = J_1 J_2$.

Remark

$\Longrightarrow$ (M, J_α, g) is (pseudo-) Kähler for $\alpha = 1, 2, 3$.

Example

$\mathbb{H}^n = \mathbb{R}^{4n}$ with the Euclidean scalar product $\langle \cdot, \cdot \rangle$ and $J_1 = L_i$, $J_2 = L_j$, $J_3 = L_k$ is a hyper-Kähler manifold.

Quaternionic Kähler manifolds I

Definition

(i) A quaternionic structure on a vector space V is a subspace $Q \subset \mathrm{End}(V)$ spanned by three pairwise anticomm. cx. structures I, J, K s.t. $K = IJ$.

(ii) An almost quaternionic structure on a manifold M is a subbundle $Q \subset \mathrm{End}(TM)$ such that Q_p is a quaternionic structure on T_pM for all p.
The bundle Q is called a quaternionic structure if it is parallel for some torsion-free connection.

(iii) Let M be a mf. of dim > 4. A quaternionic Kähler structure on M is a pair (g, Q) consisting of a Riem. metric g and a parallel quaternionic structure $Q \subset \mathfrak{so}(TM)$. The triple (M, g, Q) is called a quaternionic Kähler (QK) manifold.

Remark

If dim $M = 4$ in (iii), one has to require in addition $Q \cdot R = 0$, which is automatic in higher dimensions.

Quaternionic Kähler manifolds II

Fundamental fact

Quaternionic Kähler manifolds are Einstein. $\implies$ 3 cases: $Ric = 0$, $Ric > 0$, $Ric < 0$.

Relevance to scalar geometry of $N = 2$ theories

Hyper-Kähler mfs. = scalar mfs. of hypermultiplets, Quaternionic Kähler mfs. (of $Ric < 0$) = scalar mfs. of supergravity coupled hypermultiplets [BW].

Examples

$Ric = 0$: Ricci-flat s.c. QK mfs. are HK.

$Ric > 0$: Only known examples of complete QK mfs. of $Ric > 0$ are the Wolf spaces = QK symmetric spaces of compact type (described below).

- Simplest example: $\mathbb{H}P^n$.

Quaternionic Kähler manifolds III

Examples continued

Ric < 0 : Known complete QK mfs. of *Ric* < 0:

- QK symm. spaces of noncp. type (dual to Wolf spaces), such as $\mathbb{H}H^n$.
- Loc. symm. QK mfs. (including compact examples).
- Alekseevsky spaces (homog. including nonsymm. examples).
- Deformations of $\mathbb{H}H^n$, see [L].
- New explicit examples obtained using results explained in the next lectures.

Quaternionic Kähler manifolds IV:

The Wolf spaces can be obtained as follows:

- Let G be a cp. s.c. simple Lie group and $\mathfrak{h} \subset \mathfrak{g} = \mathrm{Lie}G$ a Cartan subalgebra,
- μ the highest root w.r.t. some system of simple roots and $s_\mu^{\mathbb{C}} = \mathrm{span}\{H_\mu, E_{\pm\mu}\} \subset \mathfrak{g}^{\mathbb{C}}$ the corresponding 3-dim. subalg.
- $H_\mu \in i\mathfrak{h}$ is normalized such that $[H_\mu, E_{\pm\mu}] = \pm 2E_{\pm\mu}$. Then ad_{H_μ} has eigenvalues $0, \pm1, \pm2$ and defines a grading

  $$\mathfrak{g}^{\mathbb{C}} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2,$$

  where $\mathfrak{g}_{\pm2} = \mathbb{C}E_{\pm\mu}$ and $\mathfrak{g}_0 = \mathbb{C}H_\mu \oplus Z_{\mathfrak{g}^{\mathbb{C}}}(s_\mu^{\mathbb{C}})$.
- Put $s_\mu := \mathfrak{g} \cap s_\mu^{\mathbb{C}}$, $\mathfrak{k} := \mathfrak{g} \cap \sum_{i=0,\pm2} \mathfrak{g}_i = s_\mu \oplus Z_{\mathfrak{g}}(s_\mu) = N_{\mathfrak{g}}(s_\mu)$, $\mathfrak{m} := \mathfrak{g} \cap \sum_{i=\pm1} \mathfrak{g}_i$.
- Then $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$ is a symmetric decomposition, which defines a cp. QK symm. space $M = G/K$, $K = N_G(s_\mu)$.

Quaternionic Kähler manifolds V:

Remarks

- The invariant quat. structure Q of the Wolf spaces is defined at $o = eK \in M = G/K$ as the image of $s_\mu \cong \mathrm{sp}(1)$ under the adjoint action on $\mathfrak{m} \cong T_o M$.
- It is skew-symmetric w.r.t. the inv. Riem. metric of M. The invariance of Q follows from $K = N_G(s_\mu)$ and implies that Q is parallel,
- since the holonomy group of M is identified with the isotropy group
$$\mathrm{Hol} = \mathrm{Ad}_K|_{\mathfrak{m}},$$
by the Ambrose-Singer theorem.
- The noncompact duals $M^{nc} = G^{nc}/K$ are obtained from the symmetric decomposition $\mathfrak{g}^{nc} = \mathfrak{k} + i\mathfrak{m}$.

Quaternionic Kähler manifolds VI:

The list of Wolf spaces $M = G/K$

- Classical:

$$\frac{\mathrm{Sp}(n+1)}{\mathrm{Sp}(n)\mathrm{Sp}(1)} = \mathbb{H}P^n = \mathrm{Gr}_1(\mathbb{H}^{n+1}),$$

$$\frac{\mathrm{SU}(n+2)}{\mathrm{S}(\mathrm{U}(n)\mathrm{U}(2))} = \mathrm{Gr}_2(\mathbb{C}^{n+2}),$$

$$\frac{\mathrm{SO}(4n+4)}{\mathrm{SO}(4n)\mathrm{SO}(4)} = \mathrm{Gr}_4(\mathbb{R}^{4n+4}),$$

- Exceptional:

$$\mathrm{G}_2/\mathrm{SO}(4), \quad \mathrm{F}_4/(\mathrm{Sp}(3)\mathrm{Sp}(1)),$$
$$\mathrm{E}_6/(\mathrm{SU}(6)\mathrm{Sp}(1)), \quad \mathrm{E}_7/(\mathrm{Spin}(12)\mathrm{Sp}(1)), \quad \mathrm{E}_8/(\mathrm{E}_7\mathrm{Sp}(1)).$$