A new mass and geometric inequalities

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Positive mass theorem (Schoen-Yau, Witten, $\cdots$)

Any asymptotically flat manifold M^n with a suitable decay order and with a dominant energy condition

$$R \geq 0$$

has the nonnegative ADM mass.

$$m_1 = m_{ADM} \geq 0.$$

Moreover, "$=$" holds $\Leftrightarrow$ $M^n = \mathbb{R}^n$.

Penrose inequality (Huisken-Ilmanen, Bray, Bray-Lee, $\cdots$)

$$m_1 = m_{ADM} \geq \frac{1}{2}\left(\frac{|\Sigma|}{\omega_{n-1}}\right)^{\frac{n-2}{n-1}},$$

where Σ is an outermost minimizing horizon and $|\Sigma|$ denotes the area of Σ.

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ADM mass

Aasymptotically flat (AF) of decay order τ if there is a compact set K such that $M \setminus K$ is diffeomorphic to $\mathbb{R}^n \setminus B_R(0)$

$$g_{ij} = \delta_{ij} + \sigma_{ij},$$

$$|\sigma_{ij}| + r|\partial\sigma_{ij}| + r^2|\partial^2\sigma_{ij}| = O(r^{-\tau})$$

ADM mass (Arnowitt-Deser-Misner):

$$m_1(g) := m_{ADM} := \frac{1}{2(n-1)\omega_{n-1}} \lim_{r\to\infty} \int_{S_r} (g_{ij,i} - g_{ii,j})\nu_j dS,$$

Bartnik: m_{AMD} is well-defined and a geometric invariant, if

$$\tau > \frac{n-2}{2} \quad \text{and} \quad R \in L^1(M).$$

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$$R(g) = \partial_j(g_{ij,i} - g_{ii,j}) + O(r^{-2\tau-2}).$$

From this expression one can check that

$$\lim_{r\to\infty} \int_{S_r} (g_{ij,i} - g_{ii,j})\nu_j dS,$$

is well defined, provided that $\tau > \frac{n-2}{2}$ and R is integrable.

Question: Can we find a similar invaraint, or mass, for AF manifolds with slower decay?

Yes, with the Gauss-Bonnet curvature, or Lovelock curvature.

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Generalized scalar curvature

Lovelock curvature or Gauss-Bonnet curvature:

$$L_k := \frac{1}{2^k}\delta^{i_1 i_2 \cdots i_{2k-1} i_{2k}}_{j_1 j_2 \cdots j_{2k-1} j_{2k}} R_{i_1 i_2}{}^{j_1 j_2} \cdots R_{i_{2k-1} i_{2k}}{}^{j_{2k-1} j_{2k}},$$

$L_1 = R$

$L_2 = |Riem|^2 - 4|Ric|^2 + R^2 = |W|^2 + 8(n-2)(n-3)\sigma_2.$

$L_{\frac{n}{2}}$ is the Pfafian, i.e., Euler-density.

σ_2 and σ_k are scalar type curvatures considered in σ_k-Yamabe problem by Viaclovsky, Chang-Gursky-Yang, Ge, Guan, W., Li, Nguyen, Sheng, Trudinger, Wang, $\cdots$.

We define a mass by using L_k for asymptotically flat manifolds with decay order

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Key Observation:

$$L_k = P_{(k)}^{stjl} R_{stjl},$$

$$P_{(k)}^{stlj} := \frac{1}{2^k} \delta^{i_1 i_2 \cdots i_{2k-3} i_{2k-2} st}_{j_1 j_2 \cdots j_{2k-3} j_{2k-2} j_{2k-1} j_{2k}} R_{i_1 i_2}{}^{j_1 j_2} \cdots R_{i_{2k-3} i_{2k-2}}{}^{j_{2k-3} j_{2k-2}} g^{j_{2k-1} l} g^{j_{2k} j}.$$

The P curvature has the same symmetry and antisymmetry and a crucial property

$$\nabla_i P^{ijlk} = 0$$

$$P_{(1)}^{ijlk} = g^{ik} g^{jl} - g^{il} g^{jk}.$$

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$$m_k(g) := m_{GBC}(g) = c_k(n) \lim_{r\to\infty} \int_{S_r} P^{ijkl} \partial_l g_{jk} \nu_i dS,$$

Theorem (Ge, W., Wu)

Suppose that (M^n, g) $(k < \frac{n}{2})$ is AF of decay order $\tau > \frac{n-2k}{k+1}$ and L_k is integrable on (M^n, g). Then the Gauss-Bonnet-Chern mass m_k is well-defined and invariant.

$$L_k = c(n,k)\partial_i \left(g_{jk,l} P^{ijkl} \right) + O(r^{-(k+1)\tau - 2k})$$

Li-Nguyen had a similar mass

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\begin{aligned}
L_k &= R_{ijkl}P^{ijkl} = g_{km}R^m_{ijl}P^{ijkl} \\
&= g_{km}(\partial_i\Gamma^m_{jl} - \partial_j\Gamma^m_{il})P^{ijkl} + O(r^{-2k-(k+1)\tau}) \\
&= g_{km}\left[\nabla_i(\Gamma^m_{jl}P^{ijkl}) - \nabla_j(\Gamma^m_{il}P^{ijkl})\right] + O(r^{-2k-(k+1)\tau}) \\
&= \frac{1}{2}\nabla_i\left[(g_{jk,l} + g_{kl,j} - g_{jl,k})P^{ijkl}\right] \\
&\quad -\frac{1}{2}\nabla_j\left[(g_{ik,l} + g_{kl,i} - g_{il,k})P^{ijkl}\right] + O(r^{-2k-(k+1)\tau}) \\
&= 2\nabla_i\left(g_{jk,l}P^{ijkl}\right) + O(r^{-2k-(k+1)\tau}) \\
&= 2\partial_i\left(g_{jk,l}P^{ijkl}\right) + O(r^{-2k-(k+1)\tau}),
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Positive mass theorem

Positive mass theorem is true for m_k, if
(1) $(\mathbb{R}^n, e^{2u}|dx|^2)$ (Ge, W. Wu, to appear in IMRN)
(2) graphical AF manifolds (Ge, W., Wu)

Theorem (Positive Mass Theorem (Ge, W., Wu))

Let $(M^n, g) = (\mathbb{R}^n, \delta + df \otimes df)$ and $L_k \in L^1(M)$, then

$$m_k = \frac{c_k(n)}{2} \int_{M^n} \frac{L_k}{\sqrt{1 + |\nabla f|^2}} dV_g,$$

In particular, $L_k \geq 0$ yields $m_k \geq 0$.

$k = 1$, Lam (2010), de Lima-Girao, Huang-Wu.

Key Lemma

$$L_k(g) = c(n)\partial_i(P^{ijkl}\partial_l g_{jk}).$$

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Theorem (Penrose Inequality ($k=1$ Lam, $k\geq 2$ Ge-W.-Wu))

$\Omega\subset\mathbb{R}^n$, $\Sigma=\partial\Omega$. $f:\mathbb{R}^n\setminus\Omega\to\mathbb{R}$, $(M,g)=(\mathbb{R}^n\backslash\Omega,\delta+df\times df)$. *$\Sigma$ is in a level set of f and $|\nabla f(x)|\to\infty$ as $x\to\Sigma$.* *Then*

$$m_k=c_k(n)\int_{M^n}\frac{L_k}{\sqrt{1+|\nabla f|^2}}dV_g+c(n)\int_\Sigma H_{2k-1}$$

In particular, if $L_k\geq 0$ (dominant energy condition) holds, then the Alexandrov-Fenchel inequality yields a Penrose inequality

$$m_2\geq\frac{1}{4}\left(\frac{\int_\Sigma R_\Sigma}{(n-1)(n-2)\omega_{n-1}}\right)^{\frac{n-4}{n-3}}\geq\frac{1}{4}\left(\frac{|\Sigma|}{\omega_{n-1}}\right)^{\frac{n-4}{n-1}}.$$

Alexandrov-Fenchel inequality in $\mathbb{R}^n$ For convex domains

$$\int_\Sigma H_k \geq \omega_{n-1} \left(\frac{1}{\omega_{n-1}} \int_\Sigma H_j \right)^{\frac{n-1-k}{n-1-j}}, \quad 0 \leq j < k \leq n-1,$$

non-convex case: Guan-Li, Huisken, Chang-Yi Wang

Penrose Inequality for AF graphs

$$m_k = m_{GBC} \geq \frac{1}{2^k} \left(\frac{|\Sigma|}{\omega_{n-1}} \right)^{\frac{n-2k}{n-1}}.$$

Optimality: The generalized anti-de Sitter Schwarzschild space-time is given by

$$(1 - \frac{2m}{\rho^{\frac{n}{k}-2}})^{-1} d\rho^2 + \rho^2 g_{\mathbb{S}^{n-1}},$$

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$$\mathbb{H}^n, b = dr^2 + \sinh^2 r g_{\mathbb{S}^{n-1}} = \frac{1}{1+\rho^2} d\rho^2 + \rho^2 g_{\mathbb{S}^{n-1}}$$

$$\mathbb{N}_b := \{V \in C^\infty(\mathbb{H}^n) | \mathrm{Hess}^b V = Vb\}.$$

$\gamma = -V^2 dt^2 + b$ is a static solution of the Einstein equation $Ric(\gamma) + n\gamma = 0$.

$$\dim \mathbb{N}_b = n+1$$

$$V_{(0)} = \cosh r, \; V_{(1)} = x^1 \sinh r, \; \cdots, \; V_{(n)} = x^n \sinh r,$$

where r is the hyperbolic distance from an arbitrary fixed point on $\mathbb{H}^n$ and $x^1, x^2, \cdots, x^n$ are the coordinate functions restricted to $\mathbb{S}^{n-1} \subset \mathbb{R}^n$. We equip the vector space $\mathbb{N}_b$ with a Lorentz metric

$$\eta(V_{(0)}, V_{(0)}) = 1, \qquad \text{and} \quad \eta(V_{(i)}, V_{(i)}) = -1 \quad \text{for} \quad i = 1, \cdots, n.$$

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$\dim \mathbb{N}_b = n + 1$

$$V_{(0)} = \cosh r,\ V_{(1)} = x^1 \sinh r,\ \cdots,\ V_{(n)} = x^n \sinh r,$$

where r is the hyperbolic distance from an arbitrary fixed point on $\mathbb{H}^n$ and $x^1, x^2, \cdots, x^n$ are the coordinate functions restricted to $\mathbb{S}^{n-1} \subset \mathbb{R}^n$. We equip the vector space $\mathbb{N}_b$ with a Lorentz metric

$$\eta(V_{(0)}, V_{(0)}) = 1, \qquad \text{and} \quad \eta(V_{(i)}, V_{(i)}) = -1 \quad \text{for} \quad i = 1, \cdots, n.$$

$$\mathbb{H}^n, b = dr^2 + \sinh^2 r g_{\mathbb{S}^{n-1}} = \frac{1}{1+\rho^2} d\rho^2 + \rho^2 g_{\mathbb{S}^{n-1}}$$

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Hyperbolic Gauss-Bonnet-Chern mass

Define a new four-tensor

$$\widetilde{\mathrm{Riem}}_{ijsl}(g) = \tilde{R}_{ijsl}(g) := R_{ijsl}(g) + g_{is}g_{jl} - g_{il}g_{js}$$

and a new Gauss-Bonnet curvature

$$\tilde{L}_k := \frac{1}{2^k}\delta^{i_1 i_2 \cdots i_{2k-1} i_{2k}}_{j_1 j_2 \cdots j_{2k-1} j_{2k}} \tilde{R}_{i_1 i_2}{}^{j_1 j_2} \cdots \tilde{R}_{i_{2k-1} i_{2k}}{}^{j_{2k-1} j_{2k}} = \tilde{R}_{stjl}\tilde{P}^{stjl}_{(k)},$$

$$\nabla_s \tilde{P}^{stjl}_{(k)} = 0.$$

Hyperbolic Gauss-Bonnet-Chern mass

$$H_k^\Phi(V) = \lim_{r\to\infty}\int_{S_r}\left(\left(V\bar\nabla_l e_{js} - e_{js}\bar\nabla_l V\right)\tilde P_{(k)}^{ijsl}\right)\nu_i d\mu$$

Theorem (Ge-W.-Wu)

Suppose $(M^n, g)(2k \le n)$ is an asymptotically hyperbolic manifold of decay order $\tau > \frac{n}{k+1}$ and for $V \in \mathbb{N}_b$, $V\tilde L_k \in L^1$, then the mass functional $H_k^\Phi(V)$ is well-defined.

$k = 1$, X. Wang, Chruściel-Herzlich, Zhang

$$V\tilde L_k = 2\bar\nabla_i\left(\left(V\bar\nabla_l e_{js} - e_{js}\bar\nabla_l V\right)\tilde P^{ijsl}\right) + 2(\bar\nabla_i\bar\nabla_l V - V b_{il})e_{js}\tilde P^{ijsl} + O(e^{(-(k+1)\tau}$$

Hyperbolic GBC mass: If $H_k^\Phi(V) > 0\ \forall V$,

$$m_k^{\mathbb{H}} := c(n,k)\inf_{\mathbb{N}_b\cap\{V>0,\eta(V,V)=1\}} H_k^\Phi(V)$$

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Penrose Inequality for AH graphs

Theorem (Penrose Inequality (Ge-W.-Wu))

$k \geq 2$. If $f : \mathbb{H}^n \setminus \Omega \to \mathbb{R}$ with $(M^n, g) = (\mathbb{H}^n \setminus \Omega, b + V^2 df \otimes df)$ is AH of decay order $\tau > \frac{n}{k+1}$ and $V\tilde{L}_k \in L^1$. Assume that $\Sigma = \partial\Omega$ is in a level set of f and $|\bar{\nabla} f(x)| \to \infty$ as $x \to \Sigma$.

$$m_k^{\mathbb{H}} = c(n,k)\left(\frac{1}{2}\int_{M^n} \frac{V\tilde{L}_k}{\sqrt{1+V^2|\bar{\nabla} f|^2}} dV_g + \frac{(2k-1)!}{2}\int_\Sigma V H_{2k-1} d\mu\right).$$

$$m_k^{\mathbb{H}} \geq \frac{1}{2^k}\left(\left(\frac{|\Sigma|}{\omega_{n-1}}\right)^{\frac{n}{k(n-1)}} + \left(\frac{|\Sigma|}{\omega_{n-1}}\right)^{\frac{n-2k}{k(n-1)}}\right)^k,$$

if $\tilde{L}_k \geq 0$ and $\Sigma \subset \mathbb{H}^n$ is horospherical convex. Moreover, equality is achieved by an anti-de Sitter Schwarzschild type metric.

$k = 1$, Dahl-Gicquaud-Sakovich, de Lima and Girão

Theorem (Ge-W.-Wu)

Let Σ be a horospherical convex hypersurface in $\mathbb{H}^n$

$$\int_\Sigma V H_{2k+1} d\mu \geq \omega_{n-1} \left(\left(\frac{|\Sigma|}{\omega_{n-1}} \right)^{\frac{n}{(k+1)(n-1)}} + \left(\frac{|\Sigma|}{\omega_{n-1}} \right)^{\frac{n-2k-2}{(k+1)(n-1)}} \right)^{k+1}.$$

Equality holds if and only if Σ is a centered geodesic sphere in $\mathbb{H}^n$.

$k = 1$ de Lima-Girao

A similar inequality by Brendle-Hung-Wang

Ideas: Inverse curvature flow by Gerhardt, Heintze-Karcher type inequality of Brendle, optimal geometric inequalities on $\mathbb{S}^{n-1}$ of Guan-W. and Alexandrov-Fenchel inequalites in $\mathbb{H}^n$:

Alexandrov-Fenchel inequality in $\mathbb{H}^n$

Isoperimetric inequality in $\mathbb{H}^n$ by Schmidt (1940)
$n = 2$. $L^2 \geq 4\pi A + A^2$

Theorem (Ge-W.-Wu, Ge-W.-Wu-Xia)

Let $1 \leq k \leq n-1$. Any horospherical convex hypersurface Σ in $\mathbb{H}^n$ satisfies

$$\int_\Sigma H_k d\mu \geq C_{n-1}^k \omega_{n-1} \left\{ \left(\frac{|\Sigma|}{\omega_{n-1}} \right)^{\frac{2}{k}} + \left(\frac{|\Sigma|}{\omega_{n-1}} \right)^{\frac{2}{k} \frac{(n-k-1)}{n-1}} \right\}^{\frac{k}{2}}.$$

Equality holds if and only if Σ is a geodesic sphere.

$k = 2$ Li-Wei-Xiong

It solves a problem in integral geometry in $\mathbb{H}^n$ proposed by Gao-Hug-Schneider, at least in the case of horospherical convex.

Mass for ALH

$P_\kappa = I_\kappa \times N$, where $I_{-1} = (1, +\infty)$ and $I_0 = I_1 = (0, \infty)$ endowed with the warped product metric

$$b_\kappa = \frac{ds^2}{V_\kappa^2(\rho)} + \rho^2 g_N, \quad \rho \in I_\kappa, \quad \text{and} \quad V_\kappa(\rho) = \sqrt{\rho^2 + \kappa}.$$

$$m_{(M,g)} = c_n \lim_{r\to\infty} \int_{N_r} \Big(V_\kappa(\operatorname{div}^{b_\kappa} e - d \operatorname{tr}^{b_\kappa} e) + (\operatorname{tr}^{b_\kappa} e - e(\nabla^{b_\kappa} V_\kappa, \cdot) \Big) \nu d\mu,$$

If $\kappa = -1$, the parameter m can be negative. In fact, m belongs to the following interval

$$m \in [m_c, +\infty) \quad \text{and} \quad m_c = -\frac{(n-2)^{\frac{n-2}{2}}}{n^{\frac{n}{2}}}.$$

Penrose Inequality for ALH graphs

Conjecture (Chrusciel-Simon)

Let (M, g) be a ALH manifold with an outermost minimal horizon Σ. Then the mass

$$m \geq \frac{1}{2}\left(\left(\frac{|\Sigma|}{\vartheta_{n-1}}\right)^{\frac{n}{n-1}} + \kappa\left(\frac{|\Sigma|}{\vartheta_{n-1}}\right)^{\frac{n-2}{n-1}}\right),$$

provided that M satisfies the dominant condition

$$R_g + n(n-1) \geq 0.$$

Moreover, equality holds if and only if (M, g) is a Kottler space.

- $n = 3$ Lee-Neves, by using the inverse mean curvature flow of Huisken-Ilmanen
- Arbitrary n, but for graphs Ge-W.-Wu-Xia by using

Theorem (Ge-W.-Wu-Xia)

Let Σ be a compact embedded hypersurface which is star-shaped with positive mean curvature in $P_{\kappa,m} = (\rho_{\kappa,m}, \infty) \times N^{n-1}$, then we have

$$\begin{aligned}\int_\Sigma V_{\kappa,m} H d\mu \;\; &\geq \;\; (n-1)\vartheta_{n-1}\left(\left(\frac{|\Sigma|}{\vartheta_{n-1}}\right)^{\frac{n}{n-1}} - \left(\frac{|\partial M|}{\vartheta_{n-1}}\right)^{\frac{n}{n-1}}\right) \\ &\quad + (n-1)\kappa\vartheta_{n-1}\left(\left(\frac{|\Sigma|}{\vartheta_{n-1}}\right)^{\frac{n-2}{n-1}} - \left(\frac{|\partial M|}{\vartheta_{n-1}}\right)^{\frac{n-2}{n-1}}\right),\end{aligned}$$

where $\partial M = \{\rho_{\kappa,m}\} \times N$. Equality holds if and only if Σ is a slice.

Proof's ideas of AF inequalities

1. Flows.

a) Inverse curvature flow (Q. Ding, Gerhardt, Brendle-Hung-Wang

b) Curvature flow preserving ... (Cabezas-Rivas, Miquel, Makowski, W.-Xia)

c) Flow of Guan-Li

2. Functional.

a) $\int H_k$

b) Quermassintergral

c) $\int L_k$

3. Monotonicty

$$\frac{\int_\Sigma L_k d\mu(g)}{|\Sigma|^{\frac{n-1-2k}{n-1}}} \geq C_{n-1}^{2k}(2k)!\omega_{n-1}^{\frac{2k}{n-1}}$$

Thanks for your attention!