Rigidity results for spin manifolds with foliated boundary

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Riemannian flow

- Let (M^{n+1}, g) be a Riemannian manifold endowed with a Riemannian flow $\mathcal{F}$ given by a unit vector field ξ. That is, the vector ξ defines a 1-dimensional foliation on M by its integral curves satisfying the rule

$$(\mathcal{L}_\xi g)(Z, W) = 0$$

for all Z, W orthogonal to ξ.

- Equivalently, this means that the endomorphism $h = \nabla^M \xi : \xi^\perp \to \xi^\perp$, called the O'Neill tensor, is a skew-symmetric tensor field.

Transversal Levi-Civita connection

- There exists a unique metric connection on the normal bundle $Q = \xi^\perp$ given by

$$\nabla_X Z = \begin{cases} \pi([X, Z]) & \text{for } X = \xi \\ \pi(\nabla^M_X Z) & \text{for } X \perp \xi \end{cases}$$

where $Z \in \Gamma(Q)$ and $\pi : TM \longrightarrow Q$ is the orthogonal projection.

- Basic Property: $\xi \lrcorner R^\nabla = 0$. Therefore, one may define $\mathrm{Ric}^\nabla, \mathrm{Scal}^\nabla, \ldots$

Gauss-type formulas

- We have the Gauss-type formulas: For all $Z, W \in \Gamma(Q)$

$$\begin{cases} \nabla^M_\xi Z &= \nabla_\xi Z + h(Z) - g(Z,\kappa)\xi \\ \nabla^M_Z W &= \nabla_Z W - g(h(Z), W)\xi \end{cases}$$

where $\kappa := \nabla^M_\xi \xi$ is the mean curvature of the flow.

Spin Riemannian flow

- Assume that M is a spin manifold. As we have the orthogonal splitting $TM = \mathbb{R}\xi \oplus Q$, the normal bundle carries a spin structure (as a vector bundle) given by the pull-back of the one on M.

- We have the isomorphisms

$$\begin{cases} \Sigma M \simeq \Sigma Q & \text{if } n \text{ is even} \\ \Sigma M \simeq \Sigma Q \oplus \Sigma Q & \text{if } n \text{ is odd.} \end{cases}$$

Spinorial Gauss-type formulas

- The Clifford multiplications on M and Q are identified as: For all $Z \in \Gamma(Q), \varphi \in \Gamma(\Sigma M)$,

$$\begin{cases} Z \cdot_M \varphi = Z \cdot_Q \varphi & \text{if } n \text{ is even} \\ Z \cdot_M \xi \cdot_M \varphi = (Z \cdot_Q \oplus - Z \cdot_Q)\varphi & \text{if } n \text{ is odd.} \end{cases}$$

- We have the spinorial Gauss-type formulas on ΣM and ΣQ: For all $\varphi \in \Gamma(\Sigma M)$

$$\begin{cases} \nabla^M_\xi \varphi = \nabla_\xi \varphi + \frac{1}{2}\Omega \cdot_M \varphi + \frac{1}{2}\xi \cdot_M \kappa \cdot_M \varphi \\ \nabla^M_Z \varphi = \nabla_Z \varphi + \frac{1}{2}\xi \cdot_M h(Z) \cdot_M \varphi \end{cases}$$

where $\Omega(\cdot,\cdot) = g(h\cdot,\cdot)$ is a 2-form on $\Gamma(Q)$.

Basic Dirac operator

- The basic Dirac operator is defined on the set of basic spinors (that is, spinors constant along the leaves) as

$$D_b = \sum_{i=1}^{n} e_i \cdot_Q \nabla_{e_i} - \frac{1}{2}\kappa \cdot_Q,$$

where $\{e_i\}_{i=1,\cdots,n}$ is an orthonormal frame of $\Gamma(Q)$.

- We have the relations

$$\begin{cases} D_M = D_b - \frac{1}{2}\xi \cdot_M \Omega \cdot_M & \text{if } n \text{ is even} \\ D_M = \xi \cdot_M (D_b \oplus -D_b) - \frac{1}{2}\xi \cdot_M \Omega \cdot_M & \text{if } n \text{ is odd.} \end{cases}$$

Manifolds with boundary

- Let (N^{n+2}, g) be a Riemannian spin manifold with smooth boundary $\partial N = M$. The unit normal vector field ν induces the spin structure on N to M. In this case, the extrinsic spinor bundle $\mathbf{S} := \Sigma N|_M$ is identified with the intrinsic one ΣM for n odd or to a double copy for n even.

- We have the Gauss formula: For all $X \in \Gamma(TM), \varphi \in \Gamma(\mathbf{S})$

$$\nabla^N_X \varphi = \nabla^{\mathbf{S}}_X \varphi + \frac{1}{2} A(X) \cdot_{\mathbf{S}} \varphi,$$

where $A = -\nabla^N \nu$ is the second fundamental form of the boundary and "$\cdot_{\mathbf{S}}$" is the Clifford multiplication given by $X \cdot_{\mathbf{S}} \varphi = X \cdot \nu \cdot \varphi$.

Eigenvalue estimate

Theorem (O. Hijazi - S. Montiel, 2001)

Let M be the compact boundary of a spin manifold (N^{n+2}, g) with non-negative scalar curvature. Assume that the mean curvautre H is positive. The first non-zero eigenvalue of the Dirac operator of M satisfies

$$\lambda \geq \frac{n+1}{2} \inf_M H.$$

The equality case is realized if and only if H is constant and any eigenspinor is the restriction of a parallel spinor on N.

Direct Application : Spinorial proof of the Alexandrov theorem.

Rigidity results

- If the boundary carries a Killing spinor, under some curvature assumptions, the boundary is totally umbilical and the ambient manifold carries a parallel spinor (it is thus Ricci-flat).
  Consequence: A complete Ricci-flat Riemannian manifold of dimension at least 3, whose mean-convex boundary is isometric to the round sphere, is a flat disc [O. Hijazi-S. Montiel, 2001].
- In general, given a solution of the Dirac equation, the boundary has to be connected and the solution is the restriction of a parallel spinor.
  Consequence: If the boundary of a manifold is isometric to the round sphere with mean curvature $H \geq 1$, the manifold is isometric to the unit closed ball [S. Raulot, 2008].

Integral inequality

Theorem (O. Hijazi - S. Montiel, 2014)

Let (N, g) be a spin manifold with non-negative scalar curvature. Assume that the mean curvature of the boundary is positive. For any spinor field $\varphi \in \Gamma(\mathbf{S})$, the inequality holds

$$0 \leq \int_M \frac{1}{H}\left(|\mathbf{D_S}\varphi|^2 - \frac{(n+1)^2}{4}H^2|\varphi|^2\right)dv,$$

where dv the volume element on M and $\mathbf{D_S}$ is the Dirac operator defined on $\mathbf{S}$.

Property: $\mathbf{D_S} = \frac{n+1}{2}H - \nu \cdot D_N - \nabla^N_\nu$.

Equality case

- The equality is characterized by the existence of two parallel spinors $\psi, \theta \in \Gamma(\Sigma N)$ such that $P_+\varphi = P_+\psi$ and $P_-\varphi = P_-\theta$. The operators $P_\pm$ are the orthogonal projections onto the eigenspaces corresponding to the ± 1-eigenvalues of the endomorphism $i\nu$.
- Direct application: Shi-Tam type inequality, Positive mass theorem...

Manifolds with foliated boundary

- Let (N^{n+2}, g) be a spin manifold whose boundary M carries a Riemannian flow given by a unit vector field ξ.
- We have the isomorphisms:

$$\begin{cases} \Sigma Q \oplus \Sigma Q \simeq \Sigma M \oplus \Sigma M \simeq \mathbf{S} & \text{if } n \text{ is even} \\ \Sigma Q \oplus \Sigma Q \simeq \Sigma M \simeq \mathbf{S} & \text{if } n \text{ is odd.} \end{cases}$$

Main results

Theorem

Let N be an $(n+2)$-dimensional compact Riemannian spin manifold with non-negative scalar curvature, whose boundary hypersurface M has a positive mean curvature H and is endowed with a Riemannian flow. Assume that there exists a spinor field φ such that $D_b\varphi = \frac{n+1}{2}H_0\varphi$, where H_0 is a positive basic function. Then, we have

$$0 \leq \int_M \frac{1}{H}\left((n+1)^2 H_0^2|\varphi|^2 + |\Omega \cdot_M \varphi|^2 - (n+1)^2 H^2|\varphi|^2\right) dv.$$

Equality case

Theorem

If we assume that $g(A(\xi), \xi) \geq 0$ in the previous theorem, then equality holds in the inequality if and only if $h = 0$ (that is the flow is a local product) and $H_0 = H$. In this case, we get that $A(\xi) = 0$ and the spinors φ and $\xi \cdot \varphi$ are respectively the restrictions of parallel spinors on N if n is even, and if n is odd the spinor $\varphi + \xi \cdot_M \varphi$ is the restriction of a parallel spinor on N.

Proof of the inequality for n even

- We have:

$$\mathbf{D_S}\varphi = D_M\varphi = \frac{n+1}{2}H_0\varphi - \frac{1}{2}\xi \cdot_M \Omega \cdot_M \varphi$$

and

$$\mathbf{D_S}(\xi \cdot \varphi) = -D_M(\xi \cdot \varphi) = \frac{n+1}{2}H_0\xi \cdot \varphi - \frac{1}{2}\nu \cdot \Omega \cdot \varphi$$

- We used the fact that $D_b(\xi \cdot) = -\xi \cdot D_b$.

- By computing the norm in both equations and taking the sum, we get the result. $\square$

Characterization of the equality case

Lemma

If the equality holds, we have

$$h(X)\cdot_M\varphi+g(A(\xi),X)\frac{H_0}{H}\varphi-\frac{1}{(n+1)H}g(A(\xi),X)\xi\cdot_M\Omega\cdot_M\varphi = -\frac{1}{(n+1)H}A(X)\cdot_M\Omega\cdot_M\varphi, \tag{1}$$

for all $X\in\Gamma(TM)$.

Sketch of the proof of the lemma

- There exists two parallel spinors ψ and θ on N such that $P_+\varphi = P_+\psi$ and $P_-\varphi = P_-\theta$. Applying $\mathbf{D_S}$ on both sides, we get $P_-(\mathbf{D_S}\varphi) = \frac{n+1}{2} HP_-\psi$ and $P_+(\mathbf{D_S}\varphi) = \frac{n+1}{2} HP_+\theta$.
- The same technique can be used for the spinor field $\xi \cdot \varphi$ and two other spinor fields Ψ and Θ exists.
- Differentiating the equations $\xi \cdot P_-\theta = P_+\Psi$ and $\xi \cdot P_+\psi = P_+\Theta$ along any vector field X in $\Gamma(TM)$, we deduce the result. $\square$

Proof of the theorem

- Taking the trace of Equation (1) and multiplying the new equation by $A(\xi)\cdot_M \xi \cdot_M$, we find

$$-A(\xi)\cdot_M \kappa \cdot_M \varphi + \mathcal{B}\Omega \cdot_M \varphi + \frac{H_0}{H}|A(\xi)|^2 \xi \cdot_M \varphi$$

$$+(n+1)H_0 A(\xi)\cdot_M \varphi + \Big((n+1)H + 2g(A(\xi),\xi)\Big)\xi \cdot_M \kappa \cdot_M \varphi = 0,$$

where $\mathcal{B} = \frac{1}{(n+1)H}|A(\xi) + (n+1)H\xi|^2 \neq 0.$

- Replacing $\Omega \cdot_M \varphi$ by its value from the above equation into (1) and taking $X = \xi$, we get $H\mathcal{I}\kappa \cdot_M \varphi + H_0 \mathcal{J}\varphi = 0$.

- The terms $\mathcal{I}$ and $\mathcal{J}$ are defined by

$$\mathcal{I} := (n+1)H + g(A(\xi), \xi),$$

$$\mathcal{J} := (n+1)Hg(A(\xi), \xi) + |A(\xi)|^2.$$

- The Hermitian product by φ gives that $h = 0$ and $A\xi = 0$. Applying $\mathbf{D_S}$ on both equalities:

$$H_0 P_+ \varphi = H P_+ \theta \quad \text{and} \quad H_0 P_- \varphi = H P_- \psi$$

yields to $H_0 = H$ and $\varphi = \psi = \theta$ on M. $\square$

Rigidity results

Corollary

Let N be a compact spin Riemannian $(n+2)$-dimensional manifold with non-negative scalar curvature, whose boundary hypersurface M has positive mean curvature H and is endowed with a Riemannian flow. Assume that there exist a spinor field φ such that $D_b\varphi = \frac{n+1}{2}H_0\varphi$, where H_0 is a positive basic function with $H_0 + \frac{1}{n+1}[\frac{n}{2}]^{\frac{1}{2}}|\Omega| \leq H$. Then the vector field ξ is parallel on M and $A(\xi) = 0$. Moreover, the spinors φ and $\xi \cdot \varphi$ are respectively the restrictions of parallel spinors on N if n is even and if n is odd, the spinor $\varphi + \xi \cdot_M \varphi$ is the restriction of a parallel spinor on N.

Basic Killing spinors

Theorem

Let (N^{n+2}, g) be a spin manifold of non-negative scalar curvature with connected boundary M of positive mean curvature H. Assume that M is endowed with a minimal Riemannian flow carrying a maximal number of basic Killing spinors of constant $-\frac{1}{2}$ (resp. a maximal number of basic Killing spinors of constants $-\frac{1}{2}$ and $\frac{1}{2}$) if n is even (resp. if n is odd). If the inequality $\frac{n}{n+1} + \frac{1}{n+1}[\frac{n}{2}]^{\frac{1}{2}}|\Omega| \leq H$ holds, the boundary M is isometric to the Riemannian product $\mathbb{S}^1 \times \mathbb{S}^n$ and N is isometric to $\mathbb{S}^1 \times B$, where B is the unit ball in $\mathbb{R}^{n+1}$.

Sketch of the proof

- From the Gauss and O'Neill formulas, we deduce that $A(X) = X$ for all $X \in \Gamma(Q)$. Moreover, from the fact that $A\xi = 0$ and $h = 0$, we deduce that N is flat (maximal number of parallel spinors) and $\widetilde{M}$ is isometric to $\mathbb{R} \times \mathbb{S}^n$. Thus $M \simeq \mathbb{S}^1 \times \mathbb{S}^n$.

- The vector field ξ can be extended to a unique parallel vector field $\hat{\xi}$ on N. It is indeed a solution of the boundary problem:

$$\begin{cases} \Delta^N \hat{\omega} = 0 & \text{on } N \\ J^* \hat{\omega} = \omega, J^*(\delta^N \hat{\omega}) = 0 & \text{on } M. \end{cases}$$

The operator J^* is the restriction to the boundary.

- Let N_1 be a connected integral submanifold of $(\mathbb{R}\hat{\xi})^{\perp}$. The manifold N_1 is complete with ∂N_1 is compact and totally umbilical in N_1.
- From the rigidity result in [M. Li, 2014], we deduce that N_1 is compact. Then from [S. Raulot, 2008] we get that ∂N_1 is connected and isometric to $\mathbb{S}^n$. Therefore $N_1 \simeq B$.
- Using the Brouwer fixed-point theorem, we finish the proof. $\square$

Corollary

Let (N^{n+2}, g) be a compact spin Riemannian manifold with non-negative scalar curvature. We assume that the boundary is isometric to $\mathbb{S}^1 \times \mathbb{S}^n$ with mean curvature $H \geq \frac{n}{n+1}$. If the induced spin structure on M is the trivial one on $\mathbb{S}^1 \times \mathbb{S}^n$, then N is isometric to the product of $\mathbb{S}^1$ with the unit ball.

More details in : Rigidity results for spin manifolds with foliated boundary, arxiv:1412.1339.