A Polyakov formula for surfaces with conical singularities and angular sectors

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Conference on Differential Geometry
Notre Dame University Beirut and American Universty of Beirut, Beirut

27 April 2015

Outline

- Motivation and Introduction

  Determinant of Laplacians on smooth manifolds

- Our setting: angular sectors and surfaces with conical singularites.

  Statement of the main results

  Remarks about these statements

- Idea of the proof

  Uniform geometric setting

  Uniform analytic setting

  Differentiation

The Laplace operator on a compact manifold.

- Let (M, g) be a compact Riemannian manifold of dimension n.
- The metric in coordinates: $g = \sum_{i,j=1}^{n} g_{ij}(x) dx_i \otimes dx_j$.

Consider the Laplacian Δ_g on a function f:

$$\Delta_g f = -\mathrm{div} \nabla f = -\frac{1}{v(x)} \sum_{k,\ell} \partial_{x_k} g^{k\ell} v(x) \partial_{x_\ell} f$$

$$= -g^{k\ell} \partial_{x_k} \partial_{x_\ell} f + \text{lower order terms}$$

where $v(x) = \sqrt{\det(g_{ij}(x))}$ and $(g^{k\ell}) = ((g_{ij})^{-1})_{k,\ell}$.

Compact manifold, the spectrum of the Dirichlet Laplacian.

Let (M, g) be a Riemannian manifold with boundary.

- We look for solutions of the Dirichlet eigenvalue problem:

$$\Delta_g u = \lambda u, \qquad u|_{\partial M} = 0, \qquad u \neq 0.$$

- We know that there exist infinitely many such solutions λ_j, u_j.

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \nearrow \infty$$

- The spectrum of the Laplacian $\mathrm{Spec}(\Delta_g) = \{\lambda_j\}_{j=1}^{\infty}$.

The spectrum of the Laplacian, example.

- Let D be a disk of radius R, $D \subset \mathbb{R}^2$.
- The Euclidean metric on D in polar coordinates: $g = dr^2 + r^2 d\phi^2$.
- The Laplacian in polar coordinates is:

$$(\Delta_D f)(r,\phi) = -\left(\frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \phi^2}\right)$$

- The spectrum of the Dirichlet Laplacian is $\lambda_{0,k} = \frac{j_{0,k}^2}{R^2}$, the k-th zero of the Bessel function J_0, $k \geq 1$. $\lambda_{n,k} = \frac{j_{n,k}^2}{R^2}$, the k-th zero of the Bessel function J_n, $k, n \geq 1$ with double multliplicity.

The determinant of the Laplacian (Ray-Singer '71)

The spectral zeta function associated to Δ_g, for $\mathrm{Re}(s) > n/2$:

$$\zeta_{\Delta_g}(s) = \sum_{\lambda_j > 0} \lambda_j^{-s} = \mathrm{Tr}((\Delta_g - P)^{-s})$$

$$= \frac{1}{\Gamma(s)} \int_0^\infty (\mathrm{Tr}(e^{-t\Delta_g}) - m) t^{s-1} dt,$$

where P is the projection on $\mathrm{Ker}(\Delta_g)$, $m = \dim(\mathrm{Ker}(\Delta_g))$ and $e^{-t\Delta_g}$, for $t > 0$, is the heat semi-group.

The determinant of the Laplacian

- Using some properties of the trace of the heat operator, $\zeta_{\Delta_g}(s)$ can be extended meromorphically to $\mathbb{C}$.
- Define the regularized determinant of Δ_g as

$$\det \Delta_g = \exp(-\frac{d}{ds}\zeta_{\Delta_g}(s)\big|_{s=0}).$$

- det(Δ_g) **is completely determined by spectrum**
- It generalizes the determinant of a positive matrix; let A be such a matrix

$$\det(A) = \prod \lambda_j = \exp \sum_j \log(\lambda_j) = \exp(-\frac{d}{ds}\sum_j \lambda_j^{-s}\big|_{s=0})$$

The determinant of the Laplacian

- Example: $M = S^1 = \mathbb{R}/2\pi\mathbb{Z}$

  Eigenvalues: $\lambda_j = j^2$

  $\zeta_{S^1}(s) = 2\sum_{j\leq 1} j^{-2s}$

  $\zeta'_{S^1}(0) = 4\zeta'_R(0),$

  $\det(\Delta_{S^1}) = 4\pi^2.$

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Remark

$\det \Delta_g$ *is a spectral invariant. It makes sense to study how it varies with respect to the metric.*

- $\det(\Delta_g)$ is a global invariant, it is not local.
- However, the variation of $\det(\Delta_g)$ is local.

Extremals of determinants on surfaces

Let M be a closed surface of genus p.
Met(M): smooth metrics on M up to isomorphism.

$$\det : \mathrm{Met}(M) \to \mathbb{R}, g \mapsto \det \Delta_g.$$

$[g] = \{h \in \mathrm{Met}(M) | h = e^{2\varphi} g, \varphi \in C^\infty(M)\}$: the conformal class.

Theorem (Osgood, Phillips, Sarnak)

In each conformal class in Met(M), up to isometry, among all metrics of unit area, there exists a unique metric τ of constant curvature at which $\det(\Delta_\tau)$ attains a maximum, i.e.

$$\det(\Delta_\tau) \geq \det(\Delta_h), \forall h \in Conf_1(g)$$

Extremals of determinants on surfaces

- Analogous results for surfaces with smooth boundary (OPS)
- Planar domains of finite connectivity and smooth boundary (OPS)
- Surfaces with with asymptotic hyperbolic cusps and funnels using renormalized determinants (Albin, Aldana and Rochon)
- Higer dimensions: dimensions 3 and 4 (A. Chang, J. Qing, T. Branson, P.Gilkey, P. Yang)
- We want to study this problem for surfaces with conical singularities.

Polyakov's formula for closed surfaces

Let (M, g) be closed. Let $h = e^{2\varphi} g$, with $\varphi \in C^\infty(M)$

$$\log \det(\Delta_h) = -\frac{1}{12\pi}\int_M |\nabla_g \varphi|^2 \, dA_g - \frac{1}{6\pi}\int_M R_g \, \varphi \, dA_g + \log A_h + \log \det(\Delta_g).$$

where R_g is the Gaussian curvature of g. (Polyakov, Alvarez, OPS, and many others)
Polyakov's formula gives a link between the determinant, which is defined in terms of the spectrum, and the second order derivatives of the conformal factor. To obtain Polyakov's formula, one uses the heat equation.

Heat equation

Let (M, g) n-dim Riem. mfd. **The heat operator $e^{-t\Delta_g}$** gives the solutions to

- $(\partial_t + \Delta_g)u = 0, u(z, 0) = f(z), u(z, t)|_{\partial M} = 0, \forall t > 0$
- It is a compact, smoothing operator. It is trace class for $t > 0$.
- It has an integral kernel and $u(z, t) = (e^{-t\Delta_g} f)(z) = \int_M K_g(z, z', t) f(z') dA_g(z')$
- By Lidskii's Thm,

$$\mathrm{Tr}(e^{-t\Delta_g}) = \sum_{j=0}^{\infty} e^{-t\lambda_j} = \int_M K_g(z, z, t) dA_g(z)$$

- The spectrum of Δ_g completely determines the heat trace.

Heat invariants in dimension 2

At $t = 0$, the heat trace diverges. Asymptotic expansion in t: If M has $\dim M = 2$, $\partial M \neq \emptyset$:

$$\mathrm{Tr}(e^{-t\Delta_g}) \sim t^{-1}\sum_{j=0}^{\infty} a_j t^j + t^{-1}\sum_{j=0}^{\infty} b_j t^{j+\frac{1}{2}}.$$

Definition

The coefficients a_j, b_j are called the heat invariants.

- They are local invariants (Gilkey, Branson...)
- Osgood, Phillips, Sarnak (closed surface), Branson, Gilkey and Orsted (more general):

$$a_j(\Delta) = \int_M (j(j-1)c_j)|\nabla^{j-2}R|^2 + \text{polynomial}(R, \nabla R, \ldots \nabla^{j-3}R)dA$$

for $j \geq 3$, where R is the scalar curvature.

Heat invariants in dimension 2

Melrose (planar domains), $M = \Omega \subset \mathbb{R}^2$

$$b_{j+1} = c_{j,u} \int_0^L |\kappa^{(j)}(s)|^2 + q_j(\kappa, \ldots, \kappa^{(j-1)}) ds$$

where $\kappa(s)$ is the curvature of $\partial\Omega$.

- In particular:
  $a_0 = \frac{\text{Area}(M,g)}{4\pi}$, $b_0 = -\frac{\text{length}(\partial M,g)}{8\sqrt{\pi}}$, $a_1 = \frac{\chi(M)}{6}$.
- The spectrum determines the area, the length of the boundary, and $\chi(M)$.

Polyakov's formula for closed surfaces

To obtain Polyakov's formula, one considers the variation of the spectral zeta function. Let (M, g), and $h = e^{2\varphi} g$, let $\psi \in C^\infty(M)$,

- Let us consider $h_u = e^{2(\varphi + u\psi)} g$, so $h_0 = h$.
- $\Delta_u = \Delta_{h_u} = e^{-2(\varphi + u\psi)} \Delta_g$
- The varition of $\log(\det(\Delta_h))$ in the direction of ψ is

$$\frac{\delta}{\delta\psi} \log(\det(\Delta_h)) = -\frac{d}{du} \zeta'_{\Delta_u}(0) = -\left. \frac{d}{ds} \frac{d}{du} \zeta_u(s) \right|_{u=0, s=0},$$

we first consider $\frac{d}{du} \zeta_u(s)$, for $\mathrm{Re}(s)$ big enough. To differentiate w.r.t. s we take its meromorphic extension to $\mathbb{C}$ that is regular at $s = 0$.

Polyakov's formula for closed surfaces

Then we have

$$\begin{aligned}
\frac{d}{ds}\frac{d}{du}\zeta_u(s)\Big|_{u=0,s=0} &= \operatorname{pf}_{t=0}\operatorname{Tr}(2\psi(e^{-t\Delta_h}-P)) \\
&= \operatorname{pf}_{t=0}\int_M 2\psi(z)(K_h(z,z,t)-m)dA_h(z) \\
&= 2\int_M \psi\left(\frac{R_h}{12\pi}-\frac{1}{A_h}\right)\, dA_h
\end{aligned}$$

where $\operatorname{pf}_{t=0}$ denotes the finite part as $t \to 0$, and R_h is the Gaussian curvature of h.

The variational Polyakov formula for closed surfaces

- We call the formula

$$\frac{\delta}{\delta\psi}\log(\det(\Delta_h)) = \mathrm{pf}_{t=0}\,\mathrm{Tr}(2\psi(e^{-t\Delta_h} - P))$$

  the variational Polyakov formula. **It is a local formula!**

- The trace $\mathrm{Tr}(\psi e^{-t\Delta_h})$ has an asymptotic expansion as $t \to 0$:

$$\mathrm{Tr}(\psi e^{-t\Delta_h}) = a_0(h,\psi)t^{-1} + a_2(h,\psi) + O(t)$$

- Then $\frac{\delta}{\delta\psi}\log(\det(\Delta_h)) = 2\,(a_2(h,\psi) - \dim(\mathrm{Ker}(\Delta_h)))$

The variational Polyakov formula for compact smooth surfaces

If $\partial M \neq \emptyset$, we have $\frac{\delta}{\delta\psi}\log(\det(\Delta_h)) = \mathrm{pf}_{t=0}\,\mathrm{Tr}(2\psi e^{-t\Delta_h})$

- $\mathrm{Tr}(\psi e^{-t\Delta_h}) = a_0(h,\psi)t^{-1} + a_1(h,\psi)t^{-1/2} + a_2(h,\psi) + O(t^{1/2})$
- Then $\frac{\delta}{\delta\psi}\log(\det(\Delta_h)) = 2a_2(h,\psi)$
- In Polyakov's formula there appear additional terms coming from the geodesic curvature of the boundary.

$$\log\det(\Delta_h) - \log\det(\Delta_g) = -\frac{1}{12\pi}\int_M |\nabla_g\varphi|^2\, dA_g$$
$$-\frac{1}{6\pi}\int_M \mathrm{K}_g\ \varphi\ dA_g - \frac{1}{6\pi}\int_{\partial M}\kappa_g\varphi ds_g - \frac{1}{4\pi}\int_{\partial M}\partial_n\varphi ds_g.$$

Planar Domains

Let $\Omega \subset \mathbb{R}^2$, bounded convex domain with continuous boundary. Let Δ be the Euclidean Laplacian. The domain of the Dirichlet Laplacian is

$$\mathrm{Dom}(\Delta) = H_0^1(\Omega) \cap H^2(\Omega)$$

(Grisvard, Ladyzhenskaya-Ural'tseva).
As before, we know that there exist infinitely many solutions λ_j, u_j to the Dirichlet eigenvalue problem The spectrum of the Laplacian Spec(Δ_g),

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \nearrow \infty$$

obeys Weyl's law and the determinant can be defined as above.

Planar Domains and sectors

Let S_α be a finite area convex sectors with $\alpha \in (0, \pi)$

Theorem (Aldana, Müller, Rowlett)

The derivative of $-\log(\det(\Delta_\alpha))$ *with respect to the angle* α *is the finite part (Hadamard's partie finie) as* $t \downarrow 0$ *of the integral*

$$\mathrm{Tr}_{L^2(S_\alpha, g)}\left(2\frac{1+\log(r)}{\alpha}e^{-t\Delta_\alpha}\right) = \int_{S_\alpha} 2\frac{1+\log(r)}{\alpha} K_{S_\alpha}(t, r, \phi, r, \phi) r dr d\phi, \quad (2.1)$$

where K_{S_α} *denotes the heat kernel on* S_α*. If the radial direction is multiplied by a factor of* R, $g \to R^2 g$, *then*

$$\det(\Delta_\alpha) \mapsto R^{-2\zeta_{\Delta_\alpha}(0)} \det(\Delta_\alpha).$$

Surfaces with conical singularities, definition

A Riemannian surface (M_γ, g) has a conical singularity at the point p with "opening angle" γ if p has a neighborhood

$$\mathcal{N} \cong [0, R]_r \times S^1_\phi,$$

in which the metric is

$$g = dr^2 + r^2\gamma^2 d\phi^2,$$

where $d\phi^2$ is the standard metric on S^1 with radius one.
Consider conformal variations of g with a conical singularity at p, but the angle may vary: In $\mathcal{N}$ the variation of the conformal factor is in the direction of

$$\xi(r, \phi) = c(\phi)\log(r), \quad \text{with } c \in C^\infty(S^1).$$

Away from p, ξ is smooth. The metrics $h_u = e^{2u\xi}g$ are metrics with a conical singularity at p.

Surfaces with conical singularities

Theorem (Aldana, Müller, Rowlett)

Let (S, g) be a Riemannian surface with a conical singularity. Then the derivative of $\zeta'_{\Delta_g}(0)$ with respect to a conformal variation of the metric g in the direction of a function ξ described above is the finite part as $t \downarrow 0$ of

$$\mathrm{Tr}_{L^2(S,g)}\left(2\xi\left(e^{-t\Delta_g} - P_{\mathrm{Ker}(\Delta_g)}\right)\right),$$

where $e^{-t\Delta_g}$ denotes the heat operator for (S, g) and $P_{\mathrm{Ker}(\Delta_g)}$ denotes the projection on the kernel of Δ_g.

Remark about computations of the finite part

Recall the following facts about the asymptotic expansion of the heat trace on a surface (or domain) with continuous boundary.

- If $\partial M = \emptyset$, the heat invariants depend on the curvature and its derivatives.
- If $\partial M \neq \emptyset$, but smooth the boundary contribute to the heat invariants through the geodesic curvature of the bdy
- If ∂M has corners then there is a contribution of the corners to the heat invariants depending on the angles

  For a_2 we have (Kac, Mazzeo-Rowlett)

$$a_2 = \frac{1}{12\pi}\left(\int_\Omega K dA + \sum_j \int_{\gamma_j} \kappa ds\right) + \sum_{j=1}^{N} \frac{\pi^2 - \alpha_j^2}{24\pi\alpha_j}$$

Remark about computations of the finite part

We expect to have the same kind of contributions here since the formula is local. For the sector S_α using a parametrix we can replace the heat kernel on S_α by the h.k. of the model in the expansion of the trace in (2.1), as follows:

- The h.k. for $\mathbb{R}^2$ for the interior away from the straight edges.
- The h.k. for $\mathbb{R}^2_+$ close to the straight edges away from the corners.
- The h.k. for the unit disk close to the curved arc away from the corners.
- The h.k. for the infinite sector with opening angle $\pi/2$ close to the corners of the circular arc which meet the straight edges.
- The h.k. for the infinite sector with opening angle α close to the vertex of the sector.

Remark about computations of the finite part

- The contribution of the interior and of the boundary will be the same as for Polyakov's formula in the smooth setting.
- Then, after we have proved the theorems, we are only left to compute the constant term in the integral

$$\int_{S_\alpha} \log(r) K_{S_{\alpha,\infty}}(t,r,\phi,r,\phi)\ r dr d\phi$$

where $K_{S_{\alpha,\infty}}$ is the heat kernel on the infinite sector with opening angle α.

Idea of the proof

The proof follows several steps

- Uniform geometric setting
- Uniform analytic setting
- Differentiation of the spectral zeta function (standard).

Idea of the proof: Uniform geometric setting

We fix $R = 1$. Let S_α with $0 < \alpha < \pi$ be fixed. Let $Q = S_\beta$ with $0 < \beta$ be also fixed.

- Let $\{S_\gamma\}_\gamma$ be a family of sectors, want to compute $\frac{d}{d\gamma}\zeta_{\Delta_\gamma}(s)$
- Consider the transformation:

$$\Psi_\gamma : Q \to S_\gamma, \quad (\rho, \theta) \mapsto \left(\rho^{\gamma/\beta}, \frac{\gamma\theta}{\beta}\right) = (r, \phi)$$

- The pull-back of the Euclidean metric g on S_γ by Ψ_γ is

$$h_\gamma := \Psi_\gamma^* g = \left(\frac{\gamma}{\beta}\right)^2 \rho^{2\gamma/\beta - 2}\left(d\rho^2 + \rho^2 d\theta^2\right)$$

- Then writing $h_\gamma = e^{2\sigma_\gamma} g$ the conformal factor is

$$\sigma_\gamma(\rho, \theta) = \log\left(\frac{\gamma}{\beta}\right) + \left(\frac{\gamma}{\beta} - 1\right)\log\rho$$

Idea of the proof: Uniform geometric setting

Since we want incomplete metrics, we require that $\gamma > \beta$

- We consider $\{(Q, h_\gamma)\}_{\gamma \geq \beta}$. So h_γ represents (S_γ, g).
- The map $\Psi_\gamma^* : C_c^\infty(S_\gamma) \to C_c^\infty(S_\gamma)$ extends to the L^2 spaces.

Proposition (AMR)

For $\gamma \geq \beta$, the map Ψ_γ^* gives an equivalence between the domain of Δ_{h_γ} and the domain of the Dirichlet self-adjoint extension of Δ_γ on the sector S_γ. Moreover,

$$\Psi_\gamma^*(\mathrm{Dom}(\Delta_{S_\gamma})) = \mathrm{Dom}(\Delta_{h_\gamma}) = H^2(Q, h_\gamma) \cap H_0^1(Q, h_\gamma),$$

with $\Delta_{h_\gamma} = \Psi_\gamma^* \circ \Delta_\gamma \circ (\Psi_\gamma^*)^{-1}$.

Idea of the proof: Uniform geometric setting

For a surface with a conical singularity (M_γ), let $Q = M_\beta = (M, g_\beta)$. Define a map Ψ_γ that restricted to $\mathcal{N}$ is given by

$$\Psi_\gamma : \mathcal{N} \subseteq Q \to \mathcal{N} \subset M_\gamma, \quad (\rho, \theta) \mapsto \left(\rho^{\gamma/\beta}, \theta\right) = (r, \phi).$$

- The conformal metric h_γ restricted to $\mathcal{N}$ is
  $h_\gamma = \Psi_\gamma^* g_\gamma = e^{2\sigma_\gamma}\left(d\rho^2 + \rho^2\beta^2 d\theta^2\right)$,
  where σ_γ is the same function as in the case of the sector.
- For $\gamma \geq \beta$, Ψ_γ induces isometries Ψ_γ^* between the Sobolev spaces $H^1(Q, h_\gamma)$ and $H^1(M_\gamma, g)$, and also between
- $H^2(Q, h_\gamma)$ and $H^2(M_\gamma, g)$, $f \in H^2(Q, h_\gamma)$ if and only if $\Psi^* f \in H^2(S_\gamma, g)$.

Idea of the proof: Uniform analytic setting

We want that all the Laplace operators Δ_{h_γ} act on the same Hilbert space $L^2(Q,g)$. We need another description of the domains.

Definition

The b-vector fields on (S_γ, g), are defined as

$$\mathcal{V}_b := C^\infty \text{ span of } \{r\partial_r, \partial_\phi\}.$$

For $m \in \mathbb{N}$, the b-Sobolev space

$$H_b^m := \{f | V_1 \dots V_j f \in L^2(S_\gamma, g) \, \forall j \leq m, \; V_1, \dots, V_j \in \mathcal{V}_b\},$$

and $H_b^0 = L^2(S,g)$. The weighted b-Sobolev spaces are

$$r^x H_b^m = \{f : \exists v \in H_b^m, \quad f = r^x v\}.$$

Idea of the proof: Uniform analytic setting

Proposition (Mazzeo; Gil, Kreine, Medoza; and other authors)

The domain of the Dirichlet Laplacian Δ_{S_γ} on S_γ is

$$\mathrm{Dom}(\Delta_{S_\gamma}) = r^2 H_b^2 \cap H_0^1.$$

The domain of the Friedrichs extension of the Laplacian on M_γ with radial coordinate r near the singularity is

$$\mathrm{Dom}(\Delta_M) = \mathbb{R} + r^2 H_b^2 = \{u : \exists u_0 \in \mathbb{R},\ v \in r^2 H_b^2,\ u = u_0 + v\}.$$

Example

Let $\psi(r, \phi) = r^x \sin(k\pi\phi/\gamma)$. Then $(r\partial_r)\psi$, $(r\partial_r)^2\psi \in r^2 H_b^2(S_\gamma)$ if and only if $x > 1$.

Idea of the proof: Uniform analytic setting

In order to make $\{\Delta_{h_\gamma}\}_\gamma$ act on the same L^2 space we consinder the maps:

- On sector (Q, h_γ)

  $\Phi_\gamma : L^2(Q, dA_{h_\gamma}) \to L^2(Q, dA), \quad f \mapsto e^{\sigma_\gamma} f = \frac{\gamma}{\beta}\rho^{\gamma/\beta - 1} f$

  $\Phi_\gamma^{-1} : L^2(Q, dA) \to L^2(Q, dA_{h_\gamma}), \quad f \mapsto e^{-\sigma_\gamma} f = \frac{\beta}{\gamma}\rho^{-\gamma/\beta + 1} f$

  Each Φ_γ is an isometry of $L^2(Q, dA_{h_\gamma})$ and $L^2(Q, dA)$.
- On surfaces (Q, h_γ) the map Φ_γ is defined in slightly differently.

  Let $\mathcal{D}(Q, dA_{h_\gamma})$ be the closure of the orthogonal complement of $\mathbb{R}$ (the constant functions) in $L^2(Q, dA_{h_\gamma})$.

  $\Phi_\gamma : \mathbb{R} \oplus \mathcal{D}(Q, dA_{h_\gamma}) \to \mathbb{R} \oplus \mathcal{D}(Q, dA), \quad (u_0, v) \mapsto (u_0, e^{\sigma_\gamma} v)$

  Φ_γ is then an isometry from its domain onto its image.

Idea of the proof: Uniform analytic setting

Proposition (AMR)

Let $\gamma \in [\beta, \pi)$. The for S_γ we have

$$\Phi_\gamma\left(\Psi_\gamma^*\left(\mathrm{Dom}(\Delta_{S_\gamma})\right)\right) \subseteq \rho^{2\gamma/\beta} H_b^2(Q, dA) \cap H_0^1(Q, dA),$$

For a surface M_γ we have

$$\Phi_\gamma\left(\Psi_\gamma^*\left(\mathrm{Dom}(\Delta_{M_\gamma})\right)\right) \subseteq \mathbb{R} + \rho^{2\gamma/\beta} H_b^2(Q, dA_\beta),$$

Moreover, in both cases we also have the nesting of domains

$$\Phi_\gamma\left(\Psi_\gamma^*\left(\mathrm{Dom}(\Delta_{S_\gamma})\right)\right) \subset \Phi_{\gamma'}\left(\Psi_{\gamma'}^*\left(\mathrm{Dom}(\Delta_{S_{\gamma'}})\right)\right), \quad \gamma' < \gamma,$$

$$\Phi_\gamma\left(\Psi_\gamma^*\left(\mathrm{Dom}(\Delta_{M_\gamma})\right)\right) \subset \Phi_{\gamma'}\left(\Psi_{\gamma'}^*\left(\mathrm{Dom}(\Delta_{M_{\gamma'}})\right)\right), \quad \gamma' < \gamma.$$

Idea of the proof: Uniform analytic setting

The family of operators we require is

$$H_\gamma := \Phi_\gamma \Psi_\gamma \circ \Delta_\gamma \circ \Psi_\gamma^{-1} \circ \Phi_\gamma^{-1} = \Phi_\gamma \circ \Delta_{h_\gamma} \circ \Phi_\gamma^{-1},$$

and we have the nesting of the domains for $\beta \leq \gamma' \leq \gamma$

$$\mathrm{Dom}(H_\gamma) \subset \mathrm{Dom}(H'_\gamma) \subset \mathrm{Dom}(\Delta) = \rho^2 H_b^2(Q) \cap H_0^1(Q),$$

where Δ is the Euclidean Laplacian on $Q = S_\beta$ in polar coordinates (ρ, θ).
In order to compute the derivative with respect to the angle at $\gamma = \alpha$, we must apply both H_γ and H_α to the elements in the domain of H_α.

Idea of the proof: Differentiation of the zeta function

Now we are ready to differentiate the spectral zeta function with respect to the angle.

- We need to differentiate the heat operator. For that we need to compute

  $e^{-tH_{\gamma_1}} - e^{-tH_{\gamma_2}}$ using Duhamel's principle.

  We need to consider the difference $H_{\gamma_1} - H_{\gamma_2}$

  and compose it with $e^{-sH_{\gamma_1}}$ or $e^{-sH_{\gamma_2}}$.
- We can only do that if the domains are suitably contained in each other.

Idea of the proof: Differentiation of the zeta function

Choice of $Q = S_\beta$

1. For $\gamma > \alpha$, set $\beta := \alpha$, so $Q = S_\alpha$.

   However $\mathrm{Dom}(H_\gamma) \subsetneq \mathrm{Dom}(\Delta)$.

   Put $\widetilde{\Delta} := \Delta\Big|_{\mathrm{Dom}(H_{\alpha+\epsilon})}, \quad \epsilon > 0,$

   for $\gamma \in [\alpha, \alpha+\epsilon]$ and $\varphi \in \mathrm{Dom}(\widetilde{\Delta})$, $(H_\gamma - \widetilde{\Delta})\varphi$.

   Thus we compute the right-sided derivative as $\gamma \downarrow \alpha$ for the operator $\widetilde{\Delta}$.

2. For $\gamma < \alpha$, the metric h_γ is complete. Then we set $\beta := \alpha/2$. So $Q = S_{\alpha/2}$.

   We take the left-sided derivative as $\gamma \uparrow \alpha$ and $\mathrm{Dom}(H_\alpha) \subset \mathrm{Dom}(H_\gamma)$ for any γ.

Computation of the constant term when $\alpha = \pi/2$

The heat kernel in the quadrant can be computed directly from the h.k of a half-line

- In polar coordinates with $u = re^{i\phi}$, $v = r'e^{i\phi'}$:

$$p_C(t,u,v) = \frac{e^{-\frac{r^2+r'^2}{4t}}}{2\pi t}\left(\cosh\left(\frac{rr'\cos(\phi'-\phi)}{2t}\right) - \cosh\left(\frac{rr'\cos(\phi'+\phi)}{2t}\right)\right)$$

- After many computations we obtain the contribution for $\alpha = \pi/2$ is

$$\mathrm{pf}_{t=0}\ \mathrm{Tr}_{L^2(S_{\pi/2},g)}\left(\mathcal{M}_{\frac{2}{\pi/2}(1+\log(r))}e^{-t\Delta_{\pi/2}}\right) = -\frac{1}{4\pi} - \frac{\gamma_e}{4\pi}.$$