A Polyakov formula for surfaces with conical singularities and angular sectors

Clara L. Aldana

University of Luxembourg.
Joint work with W. Müller and J. Rowlett
Conference on Differential Geometry
Notre Dame University Beirut and American University of Beirut, Beirut

27 April 2015
Motivation and Introduction

Determinant of Laplacians on smooth manifolds

Our setting: angular sectors and surfaces with conical singularities.

Statement of the main results

Remarks about these statements

Idea of the proof

Uniform geometric setting

Uniform analytic setting

Differentiation
Let \((M, g)\) be a compact Riemannian manifold of dimension \(n\).

The metric in coordinates: \(g = \sum_{i,j=1}^{n} g_{ij}(x) dx_i \otimes dx_j\).

Consider the Laplacian \(\Delta_g\) on a function \(f\):

\[
\Delta_g f = -\text{div} \nabla f = -\frac{1}{v(x)} \sum_{k,\ell} \partial_{x_k} g^{k\ell} v(x) \partial_{x_\ell} f
\]

\[
= -g^{k\ell} \partial_{x_k} \partial_{x_\ell} f + \text{lower order terms}
\]

where \(v(x) = \sqrt{\det(g_{ij}(x))}\) and \((g^{k\ell}) = ((g_{ij})^{-1})_{k,\ell}\).
Let \((M, g)\) be a Riemannian manifold with boundary.

- We look for solutions of the Dirichlet eigenvalue problem:
  \[
  \Delta_g u = \lambda u, \quad u|_{\partial M} = 0, \quad u \neq 0.
  \]

- We know that there exist infinitely many such solutions \(\lambda_j, u_j\).
  \[
  0 < \lambda_1 \leq \lambda_2 \leq \cdots \uparrow \infty
  \]

- The spectrum of the Laplacian \(\text{Spec}(\Delta_g) = \{\lambda_j\}_{j=1}^{\infty}\).
The spectrum of the Laplacian, example.

- Let $D$ be a disk of radius $R$, $D \subset \mathbb{R}^2$.
- The Euclidean metric on $D$ in polar coordinates:
  $$g = dr^2 + r^2 d\phi^2.$$ 
- The Laplacian in polar coordinates is:
  $$(\Delta_D f)(r, \phi) = - \left( \frac{\partial^2 f}{\partial r^2} + \frac{1}{r} \frac{\partial f}{\partial r} + \frac{1}{r^2} \frac{\partial^2 f}{\partial \phi^2} \right)$$
- The spectrum of the Dirichlet Laplacian is
  $$\lambda_{0,k} = \frac{j_{0,k}^2}{R^2}, \text{ the } k\text{-th zero of the Bessel function } J_0, \ k \geq 1.$$ 
  $$\lambda_{n,k} = \frac{j_{n,k}^2}{R^2}, \text{ the } k\text{-th zero of the Bessel function } J_n, \ k, n \geq 1$$
  with double multiplicity.
The spectral zeta function associated to $\Delta_g$, for $\text{Re}(s) > n/2$:

$$
\zeta_{\Delta_g}(s) = \sum_{\lambda_j > 0} \lambda_j^{-s} = \text{Tr}((\Delta_g - P)^{-s})
$$

$$
= \frac{1}{\Gamma(s)} \int_0^\infty (\text{Tr}(e^{-t\Delta_g}) - m) t^{s-1} dt,
$$

where $P$ is the projection on $\text{Ker}(\Delta_g)$, $m = \text{dim}(\text{Ker}(\Delta_g))$ and $e^{-t\Delta_g}$, for $t > 0$, is the heat semi-group.
The determinant of the Laplacian

- Using some properties of the trace of the heat operator, \( \zeta_{\Delta_g}(s) \) can be extended meromorphically to \( \mathbb{C} \).
- Define the regularized determinant of \( \Delta_g \) as
  \[
  \det \Delta_g = \exp(-\frac{d}{ds} \zeta_{\Delta_g}(s) \big|_{s=0}).
  \]
- \( \det(\Delta_g) \) is completely determined by spectrum
- It generalizes the determinant of a positive matrix; let \( A \) be such a matrix
  \[
  \det(A) = \prod \lambda_j = \exp \sum_j \log(\lambda_j) = \exp(-\frac{d}{ds} \sum_j \lambda_j^{-s} \big|_{s=0})
  \]
The determinant of the Laplacian

Example: $M = S^1 = \mathbb{R} / 2\pi \mathbb{Z}$
Eigenvalues: $\lambda_j = j^2$

$\zeta_{S^1}(s) = 2 \sum_{j \leq 1} j^{-2s}$

$\zeta'_{S^1}(0) = 4 \zeta'_{\mathbb{R}}(0)$,
det($\Delta_{S^1}$) = $4\pi^2$. 

Remark: $\det \Delta_g$ is a spectral invariant. It makes sense to study how it varies with respect to the metric.
$\det(\Delta_g)$ is a global invariant, it is not local.
However, the variation of $\det(\Delta_g)$ is local.
The determinant of the Laplacian

- **Example:** \( M = S^{1} = \mathbb{R} / 2\pi \mathbb{Z} \)
  - Eigenvalues: \( \lambda_j = j^2 \)
  - \( \zeta_{S^1}(s) = 2 \sum_{j \leq 1} j^{-2s} \)
  - \( \zeta'_{S^1}(0) = 4\zeta'_R(0) \)
  - \( \det(\Delta_{S^1}) = 4\pi^2 \).

**Remark**

\( \det \Delta_g \) is a spectral invariant. It makes sense to study how it varies with respect to the metric.

- \( \det(\Delta_g) \) is a global invariant, it is not local.
- However, the variation of \( \det(\Delta_g) \) is local.
Let $M$ be a closed surface of genus $p$.
Met($M$): smooth metrics on $M$ up to isomorphism.

$$\det: \text{Met}(M) \rightarrow \mathbb{R}, \ g \mapsto \det \Delta_g.$$ 

$[g] = \{h \in \text{Met}(M)|h = e^{2\varphi}g, \varphi \in C^\infty(M)\}$: the conformal class.

**Theorem (Osgood, Phillips, Sarnak)**

*In each conformal class in $\text{Met}(M)$, up to isometry, among all metrics of unit area, there exists a unique metric $\tau$ of constant curvature at which $\det(\Delta_\tau)$ attains a maximum, i.e.*

$$\det(\Delta_\tau) \geq \det(\Delta_h), \ \forall h \in \text{Conf}_1(g)$$
Analogous results for surfaces with smooth boundary (OPS)

Planar domains of finite connectivity and smooth boundary (OPS)

Surfaces with asymptotic hyperbolic cusps and funnels using renormalized determinants (Albin, Aldana and Rochon)

Higher dimensions: dimensions 3 and 4 (A. Chang, J. Qing, T. Branson, P. Gilkey, P. Yang)

We want to study this problem for surfaces with conical singularities.
Polyakov’s formula for closed surfaces

Let \((M, g)\) be closed. Let \(h = e^{2\varphi} g\), with \(\varphi \in C^\infty(M)\)

\[
\log \det(\Delta_h) = -\frac{1}{12\pi} \int_M |\nabla_g \varphi|^2 \, dA_g - \frac{1}{6\pi} \int_M R_g \, \varphi \, dA_g \\
+ \log A_h + \log \det(\Delta_g).
\]

where \(R_g\) is the Gaussian curvature of \(g\). (Polyakov, Alvarez, OPS, and many others)

Polyakov’s formula gives a link between the determinant, which is defined in terms of the spectrum, and the second order derivatives of the conformal factor. To obtain Polyakov’s formula, one uses the heat equation.
Let $(M, g)$ be an $n$-dimensional Riemannian manifold. **The heat operator** $e^{-t\Delta g}$ gives the solutions to

\begin{align*}
(\partial_t + \Delta_g)u &= 0, \quad u(z, 0) = f(z), \quad u(z, t)|_{\partial M} = 0, \quad \forall t > 0 \\
\text{It is a compact, smoothing operator. It is trace class for } t > 0. \\
\text{It has an integral kernel and} \\
u(z, t) &= (e^{-t\Delta_g} f)(z) = \int_M K_g(z, z', t) f(z') dA_g(z') \\
\text{By Lidskii’s Thm,} \\
\text{The spectrum of } \Delta_g \text{ completely determines the heat trace.}
\end{align*}
Heat invariants in dimension 2

At $t = 0$, the heat trace diverges. Asymptotic expansion in $t$: If $M$ has $\dim M = 2$, $\partial M \neq \emptyset$:

$$\text{Tr}(e^{-t\Delta_g}) \sim t^{-1} \sum_{j=0}^{\infty} a_j t^j + t^{-1} \sum_{j=0}^{\infty} b_j t^{j+\frac{1}{2}}.$$ 

Definition

The coefficients $a_j$, $b_j$ are called the heat invariants.

- They are local invariants (Gilkey, Branson...)
- Osgood, Phillips, Sarnak (closed surface), Branson, Gilkey and Orsted (more general):

$$a_j(\Delta) = \int_M (j(j-1)c_j)|\nabla^{j-2}R|^2 + \text{polynomial}(R, \nabla R, \ldots \nabla^{j-3}R) dA$$

for $j \geq 3$, where $R$ is the scalar curvature.
Heat invariants in dimension 2

Melrose (planar domains), $M = \Omega \subset \mathbb{R}^2$

$$b_{j+1} = c_{j,u} \int_0^L |\kappa^{(j)}(s)|^2 + q_j(\kappa, \ldots, \kappa^{(j-1)})ds$$

where $\kappa(s)$ is the curvature of $\partial \Omega$.

- In particular:
  $$a_0 = \frac{\text{Area}(M,g)}{4\pi}, \quad b_0 = -\frac{\text{length}(\partial M,g)}{8\sqrt{\pi}}, \quad a_1 = \frac{\chi(M)}{6}.$$

- The spectrum determines the area, the length of the boundary, and $\chi(M)$. 
To obtain Polyakov’s formula, one considers the variation of the spectral zeta function. Let \((M, g)\), and \(h = e^{2\varphi}g\), let \(\psi \in C^\infty(M)\),

- Let us consider \(h_u = e^{2(\varphi + u\psi)}g\), so \(h_0 = h\).
- \(\Delta_u = \Delta_{h_u} = e^{-2(\varphi + u\psi)}\Delta_g\)
- The variation of \(\log(\det(\Delta_h))\) in the direction of \(\psi\) is
  \[
  \frac{\delta}{\delta \psi} \log(\det(\Delta_h)) = -\frac{d}{du} \zeta'_{\Delta_u}(0) = -\frac{d}{ds} \frac{d}{du} \zeta_u(s) \bigg|_{u=0,s=0},
  \]

we first consider \(\frac{d}{du} \zeta_u(s)\), for \(\text{Re}(s)\) big enough. To differentiate w.r.t. \(s\) we take its meromorphic extension to \(\mathbb{C}\) that is regular at \(s = 0\).
Then we have

$$
\left. \frac{d}{ds} \frac{d}{du} \zeta_{u}(s) \right|_{u=0, s=0} = \text{pf}_{t=0} \, \text{Tr}(2\psi(e^{-t\Delta h} - P))
$$

$$
= \text{pf}_{t=0} \int_{M} 2\psi(z)(K_h(z, z, t) - m)\,dA_h(z)
$$

$$
= 2 \int_{M} \psi \left( \frac{R_h}{12\pi} - \frac{1}{A_h} \right) \, dA_h
$$

where \( \text{pf}_{t=0} \) denotes the finite part as \( t \to 0 \), and \( R_h \) is the Gaussian curvature of \( h \).
The variational Polyakov formula for closed surfaces

- We call the formula
  \[ \frac{\delta}{\delta \psi} \log(\det(\Delta_h)) = \text{pf}_{t=0} \text{Tr}(2\psi(e^{-t\Delta_h} - P)) \]
  the variational Polyakov formula. It is a local formula!
- The trace \( \text{Tr}(\psi e^{-t\Delta_h}) \) has an asymptotic expansion as \( t \to 0 \):
  \[ \text{Tr}(\psi e^{-t\Delta_h}) = a_0(h, \psi)t^{-1} + a_2(h, \psi) + O(t) \]
- Then \( \frac{\delta}{\delta \psi} \log(\det(\Delta_h)) = 2 \left( a_2(h, \psi) - \dim(\text{Ker}(\Delta_h)) \right) \).
The variational Polyakov formula for compact smooth surfaces

If $\partial M \neq \emptyset$, we have

$$\frac{\delta}{\delta \psi} \log(\det(\Delta_h)) = \text{pf}_{t=0} \text{Tr}(2\psi e^{-t\Delta_h})$$

- $\text{Tr}(\psi e^{-t\Delta_h}) = a_0(h, \psi)t^{-1} + a_1(h, \psi)t^{-1/2} + a_2(h, \psi) + O(t^{1/2})$
- Then $\frac{\delta}{\delta \psi} \log(\det(\Delta_h)) = 2a_2(h, \psi)$

In Polyakov’s formula there appear additional terms coming from the geodesic curvature of the boundary.

$$\log \det(\Delta_h) - \log \det(\Delta_g) = -\frac{1}{12\pi} \int_M |\nabla_g \varphi|^2 \, dA_g$$

$$- \frac{1}{6\pi} \int_M K_g \varphi \, dA_g - \frac{1}{6\pi} \int_{\partial M} \kappa_g \varphi ds_g - \frac{1}{4\pi} \int_{\partial M} \partial_n \varphi ds_g.$$
Let $\Omega \subset \mathbb{R}^2$, bounded convex domain with continuous boundary. Let $\Delta$ be the Euclidean Laplacian. The domain of the Dirichlet Laplacian is

$$\text{Dom}(\Delta) = H^1_0(\Omega) \cap H^2(\Omega)$$

(Grisvard, Ladyzhenskaya-Ural’tseva).

As before, we know that there exist infinitely many solutions $\lambda_j, u_j$ to the Dirichlet eigenvalue problem The spectrum of the Laplacian $\text{Spec}(\Delta_g)$,

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \nearrow \infty$$

obeys Weyl’s law and the determinant can be defined as above.
Planar Domains and sectors

Let $S_\alpha$ be a finite area convex sectors with $\alpha \in (0, \pi)$

**Theorem (Aldana, Müller, Rowlett)**

The derivative of $-\log(\det(\Delta_\alpha))$ with respect to the angle $\alpha$ is the finite part (Hadamard’s partie finie) as $t \downarrow 0$ of the integral

$$\text{Tr}_{L^2}(S_\alpha, g) \left( 2 \frac{1 + \log(r)}{\alpha} e^{-t\Delta_\alpha} \right)$$

$$= \int_{S_\alpha} 2 \frac{1 + \log(r)}{\alpha} K_{S_\alpha}(t, r, \phi, r, \phi) r dr d\phi, \quad (2.1)$$

where $K_{S_\alpha}$ denotes the heat kernel on $S_\alpha$. If the radial direction is multiplied by a factor of $R$, $g \to R^2 g$, then

$$\det(\Delta_\alpha) \mapsto R^{-2\zeta_{\Delta_\alpha}(0)} \det(\Delta_\alpha).$$
Surfaces with conical singularities, definition

A Riemannian surface \((M_\gamma, g)\) has a conical singularity at the point \(p\) with “opening angle” \(\gamma\) if \(p\) has a neighborhood

\[
\mathcal{N} \cong [0, R]_r \times S^1_\phi,
\]

in which the metric is

\[
g = dr^2 + r^2 \gamma^2 d\phi^2,
\]

where \(d\phi^2\) is the standard metric on \(S^1\) with radius one. Consider conformal variations of \(g\) with a conical singularity at \(p\), but the angle may vary: In \(\mathcal{N}\) the variation of the conformal factor is in the direction of

\[
\xi(r, \phi) = c(\phi) \log(r), \quad \text{with} \quad c \in C^\infty(S^1).
\]

Away from \(p\), \(\xi\) is smooth. The metrics \(h_u = e^{2u\xi}g\) are metrics with a conical singularity at \(p\).
**Theorem (Aldana, Müller, Rowlett)**

Let \((S, g)\) be a Riemannian surface with a conical singularity. Then the derivative of \(\zeta'_{\Delta_g}(0)\) with respect to a conformal variation of the metric \(g\) in the direction of a function \(\xi\) described above is the finite part as \(t \downarrow 0\) of

\[
\text{Tr}_{L^2(S,g)} \left( 2\xi \left( e^{-t\Delta_g} - P_{\text{Ker}(\Delta_g)} \right) \right),
\]

where \(e^{-t\Delta_g}\) denotes the heat operator for \((S, g)\) and \(P_{\text{Ker}(\Delta_g)}\) denotes the projection on the kernel of \(\Delta_g\).
Remark about computations of the finite part

Recall the following facts about the asymptotic expansion of the heat trace on a surface (or domain) with continuous boundary.

- If $\partial M = \emptyset$, the heat invariants depend on the curvature and its derivatives.
- If $\partial M \neq \emptyset$, but smooth the boundary contribute to the heat invariants through the geodesic curvature of the bdy
- If $\partial M$ has corners then there is a contribution of the corners to the heat invariants depending on the angles

For $a_2$ we have (Kac, Mazzeo-Rowlett)

$$a_2 = \frac{1}{12\pi} \left( \int_{\Omega} KdA + \sum_j \int_{\gamma_j} \kappa ds \right) + \sum_{j=1}^{N} \frac{\pi^2 - \alpha_j^2}{24\pi \alpha_j}$$
We expect to have the same kind of contributions here since the formula is local. For the sector $S_\alpha$ using a parametrix we can replace the heat kernel on $S_\alpha$ by the h.k. of the model in the expansion of the trace in (2.1), as follows:

- The h.k. for $\mathbb{R}^2$ for the interior away from the straight edges.
- The h.k. for $\mathbb{R}_+^2$ close to the straight edges away from the corners.
- The h.k. for the unit disk close to the curved arc away from the corners.
- The h.k. for the infinite sector with opening angle $\pi/2$ close to the corners of the circular arc which meet the straight edges.
- The h.k. for the infinite sector with opening angle $\alpha$ close to the vertex of the sector.
The contribution of the interior and of the boundary will be the same as for Polyakov’s formula in the smooth setting.

Then, after we have proved the theorems, we are only left to compute the constant term in the integral

\[ \int_{S_\alpha} \log(r)K_{S_\alpha,\infty}(t, r, \phi, r, \phi) \ rdrd\phi \]

where \( K_{S_\alpha,\infty} \) is the heat kernel on the infinite sector with opening angle \( \alpha \).
The proof follows several steps

- Uniform geometric setting
- Uniform analytic setting
- Differentiation of the spectral zeta function (standard).
We fix \( R = 1 \). Let \( S_\alpha \) with \( 0 < \alpha < \pi \) be fixed. Let \( Q = S_\beta \) with \( 0 < \beta \) be also fixed.

- Let \( \{S_\gamma\}_\gamma \) be a family of sectors, want to compute \( \frac{d}{d\gamma} \zeta_{\Delta_\gamma}(s) \)
- Consider the transformation:

\[
\Psi_\gamma : Q \to S_\gamma, \quad (\rho, \theta) \mapsto \left( \frac{\rho^{\gamma/\beta}}{\beta}, \frac{\gamma \theta}{\beta} \right) = (r, \phi)
\]

- The pull-back of the Euclidean metric \( g \) on \( S_\gamma \) by \( \Psi_\gamma \) is

\[
h_\gamma := \Psi_\gamma^* g = \left( \frac{\gamma}{\beta} \right)^2 \rho^{2\gamma/\beta - 2} \left( d\rho^2 + \rho^2 d\theta^2 \right)
\]

- Then writing \( h_\gamma = e^{2\sigma_\gamma} g \) the conformal factor is

\[
\sigma_\gamma(\rho, \theta) = \log \left( \frac{\gamma}{\beta} \right) + \left( \frac{\gamma}{\beta} - 1 \right) \log \rho
\]
Idea of the proof: Uniform geometric setting

Since we want incomplete metrics, we require that $\gamma > \beta$

- We consider $\{(Q, h_{\gamma})\}_{\gamma \geq \beta}$. So $h_{\gamma}$ represents $(S_{\gamma}, g)$.
- The map $\Psi^{*}_{\gamma} : C^{\infty}_{c}(S_{\gamma}) \rightarrow C^{\infty}_{c}(S_{\gamma})$ extends to the $L^2$ spaces.

**Proposition (AMR)**

*For $\gamma \geq \beta$, the map $\Psi^{*}_{\gamma}$ gives an equivalence between the domain of $\Delta_{h_{\gamma}}$ and the domain of the Dirichlet self-adjoint extension of $\Delta_{\gamma}$ on the sector $S_{\gamma}$. Moreover,*

\[
\Psi^{*}_{\gamma} (\text{Dom}(\Delta_{S_{\gamma}})) = \text{Dom}(\Delta_{h_{\gamma}}) = H^{2}(Q, h_{\gamma}) \cap H^{1}_{0}(Q, h_{\gamma}),
\]

*with $\Delta_{h_{\gamma}} = \Psi^{*}_{\gamma} \circ \Delta_{\gamma} \circ (\Psi^{*}_{\gamma})^{-1}$.***
For a surface with a conical singularity \((M_{\gamma})\), let \(Q = M_\beta = (M, g_\beta)\). Define a map \(\Psi_\gamma\) that restricted to \(\mathcal{N}\) is given by

\[
\Psi_\gamma : \mathcal{N} \subseteq Q \rightarrow \mathcal{N} \subset M_\gamma, \quad (\rho, \theta) \mapsto (\rho^{\gamma/\beta}, \theta) = (r, \phi).
\]

- The conformal metric \(h_\gamma\) restricted to \(\mathcal{N}\) is

\[
h_\gamma = \Psi_\gamma^* g_\gamma = e^{2\sigma_\gamma} \left( d\rho^2 + \rho^2 \beta^2 d\theta^2 \right),
\]

where \(\sigma_\gamma\) is the same function as in the case of the sector.

- For \(\gamma \geq \beta\), \(\Psi_\gamma\) induces isometries \(\Psi_\gamma^*\) between the Sobolev spaces \(H^1(Q, h_\gamma)\) and \(H^1(M_\gamma, g)\), and also between \(H^2(Q, h_\gamma)\) and \(H^2(M_\gamma, g)\), \(f \in H^2(Q, h_\gamma)\) if and only if \(\Psi_\gamma^* f \in H^2(S_\gamma, g)\).
Idea of the proof: Uniform analytic setting

We want that all the Laplace operators $\Delta_{h,\gamma}$ act on the same Hilbert space $L^2(Q, g)$. We need another description of the domains.

**Definition**

The $b$-vector fields on $(S_\gamma, g)$, are defined as

$$\mathcal{V}_b := C^\infty \text{ span of } \{r \partial_r, \partial_\phi\}.$$ 

For $m \in \mathbb{N}$, the $b$-Sobolev space

$$H^m_b := \{f | V_1 \ldots V_j f \in L^2(S_\gamma, g) \forall j \leq m, V_1, \ldots, V_j \in \mathcal{V}_b\},$$

and $H^0_b = L^2(S, g)$. The weighted $b$-Sobolev spaces are

$$r^x H^m_b = \{f : \exists v \in H^m_b, \ f = r^x v\}.$$
Motivation and Introduction
Our setting, sectors and cones
Idea of the proof

Idea of the proof: Uniform analytic setting

Proposition (Mazzeo; Gil, Kreine, Medoza; and other authors)

The domain of the Dirichlet Laplacian $\Delta_{S_\gamma}$ on $S_\gamma$ is

$$\text{Dom}(\Delta_{S_\gamma}) = r^2 H^2_b \cap H^1_0.$$ 

The domain of the Friedrichs extension of the Laplacian on $M_\gamma$ with radial coordinate $r$ near the singularity is

$$\text{Dom}(\Delta_M) = \mathbb{R} + r^2 H^2_b = \{ u : \exists u_0 \in \mathbb{R}, \, v \in r^2 H^2_b, \, u = u_0 + v \}.$$ 

Example

Let $\psi(r, \phi) = r^x \sin(k\pi\phi/\gamma)$. Then $(r \partial_r)\psi, (r \partial_r)^2 \psi \in r^2 H^2_b(S_\gamma)$ if and only if $x > 1$. 

Clara L. Aldana
A Polyakov formula for surfaces with conical singularities and
Idea of the proof: Uniform analytic setting

In order to make \( \{\Delta_{h\gamma}\}_\gamma \) act on the same \( L^2 \) space we consider the maps:

- **On sector** \((Q, h_\gamma)\)
  \[
  \Phi_\gamma : L^2(Q, dA_{h\gamma}) \to L^2(Q, dA), \quad f \mapsto e^{\sigma_\gamma} f = \frac{\gamma}{\beta} \rho^{\gamma/\beta - 1} f
  \]
  \[
  \Phi^{-1}_\gamma : L^2(Q, dA) \to L^2(Q, dA_{h_\gamma}), \quad f \mapsto e^{-\sigma_\gamma} f = \frac{\beta}{\gamma} \rho^{-\gamma/\beta + 1} f
  \]
  Each \( \Phi_\gamma \) is an isometry of \( L^2(Q, dA_{h_\gamma}) \) and \( L^2(Q, dA) \).

- **On surfaces** \((Q, h_\gamma)\) the map \( \Phi_\gamma \) is defined in slightly differently.
  Let \( \mathcal{D}(Q, dA_{h_\gamma}) \) be the closure of the orthogonal complement of \( \mathbb{R} \) (the constant functions) in \( L^2(Q, dA_{h_\gamma}) \).
  \[
  \Phi_\gamma : \mathbb{R} \oplus \mathcal{D}(Q, dA_{h_\gamma}) \to \mathbb{R} \oplus \mathcal{D}(Q, dA), \quad (u_0, v) \mapsto (u_0, e^{\sigma_\gamma} v)
  \]
  \( \Phi_\gamma \) is then an isometry from its domain onto its image.
Idea of the proof: Uniform analytic setting

**Proposition (AMR)**

Let $\gamma \in [\beta, \pi)$. The for $S_\gamma$ we have

$$\Phi_\gamma \left( \psi_\gamma^* \left( \operatorname{Dom}(\Delta S_\gamma) \right) \right) \subseteq \rho^{2\gamma/\beta} H^2_b(Q, dA) \cap H^1_0(Q, dA),$$

For a surface $M_\gamma$ we have

$$\Phi_\gamma \left( \psi_\gamma^* \left( \operatorname{Dom}(\Delta M_\gamma) \right) \right) \subseteq \mathbb{R} + \rho^{2\gamma/\beta} H^2_b(Q, dA_\beta),$$

Moreover, in both cases we also have the nesting of domains

$$\Phi_\gamma \left( \psi_\gamma^* \left( \operatorname{Dom}(\Delta S_\gamma) \right) \right) \subset \Phi_{\gamma'} \left( \psi_{\gamma'}^* \left( \operatorname{Dom}(\Delta S_{\gamma'}) \right) \right), \quad \gamma' < \gamma,$$

$$\Phi_\gamma \left( \psi_\gamma^* \left( \operatorname{Dom}(\Delta M_\gamma) \right) \right) \subset \Phi_{\gamma'} \left( \psi_{\gamma'}^* \left( \operatorname{Dom}(\Delta M_{\gamma'}) \right) \right), \quad \gamma' < \gamma.$$
Idea of the proof: Uniform analytic setting

The family of operators we require is

\[ H_\gamma := \Phi_\gamma \Psi_\gamma \circ \Delta_\gamma \circ \Psi_\gamma^{-1} \circ \Phi_\gamma^{-1} = \Phi_\gamma \circ \Delta_{h_\gamma} \circ \Phi_\gamma^{-1} , \]

and we have the nesting of the domains for \( \beta \leq \gamma' \leq \gamma \)

\[ \text{Dom}(H_\gamma) \subset \text{Dom}(H'_{\gamma'}) \subset \text{Dom}(\Delta) = \rho^2 H^2_b(Q) \cap H^1_0(Q), \]

where \( \Delta \) is the Euclidean Laplacian on \( Q = S_\beta \) in polar coordinates \((\rho, \theta)\).

In order to compute the derivative with respect to the angle at \( \gamma = \alpha \), we must apply both \( H_\gamma \) and \( H_\alpha \) to the elements in the domain of \( H_\alpha \).
Idea of the proof: Differentiation of the zeta function

Now we are ready to differentiate the spectral zeta function with respect to the angle.

- We need to differentiate the heat operator. For that we need to compute
  \[ e^{-tH_{\gamma_1}} - e^{-tH_{\gamma_2}} \]
  using Duhamel’s principle.

  We need to consider the difference \( H_{\gamma_1} - H_{\gamma_2} \)
  and compose it with \( e^{-sH_{\gamma_1}} \) or \( e^{-sH_{\gamma_2}} \).

- We can only do that if the domains are suitably contained in each other.
Idea of the proof: Differentiation of the zeta function

Choice of $Q = S_\beta$

1. For $\gamma > \alpha$, set $\beta := \alpha$, so $Q = S_\alpha$.

   However $\text{Dom}(H_\gamma) \subsetneq \text{Dom}(\Delta)$.

   Put $\tilde{\Delta} := \Delta \bigg|_{\text{Dom}(H_{\alpha+\epsilon})}$, $\epsilon > 0$,

   for $\gamma \in [\alpha, \alpha + \epsilon]$ and $\varphi \in \text{Dom}(\tilde{\Delta})$, $(H_\gamma - \tilde{\Delta})\varphi$.

   Thus we compute the right-sided derivative as $\gamma \downarrow \alpha$ for the operator $\tilde{\Delta}$.

2. For $\gamma < \alpha$, the metric $h_\gamma$ is complete. Then we set $\beta := \alpha/2$.

   So $Q = S_{\alpha/2}$.

   We take the left-sided derivative as $\gamma \uparrow \alpha$ and

   $\text{Dom}(H_\alpha) \subset \text{Dom}(H_\gamma)$ for any $\gamma$. 
Computation of the constant term when $\alpha = \pi/2$

The heat kernel in the quadrant can be computed directly from the h.k of a half-line

- In polar coordinates with $u = re^{i\phi}$, $v = r'e^{i\phi'}$:

$$p_C(t, u, v) = e^{-\frac{r^2 + r'^2}{4t}} \frac{2\pi t}{2\pi} \left( \cosh \left( \frac{rr' \cos(\phi' - \phi)}{2t} \right) - \cosh \left( \frac{rr' \cos(\phi' + \phi)}{2t} \right) \right)$$

- After many computations we obtain the contribution for $\alpha = \pi/2$ is

$$\text{pf}_{t=0} \quad \text{Tr}_{L^2(S_{\pi/2}, g)} \left( \mathcal{M} \frac{2}{\pi/2} (1+\log(r)) e^{-t\Delta_{\pi/2}} \right) = -\frac{1}{4\pi} - \frac{\gamma e}{4\pi}.$$